On the characteristic rank and cohomology of oriented Grassmann manifolds

Tomáš Rusin

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Introduction

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Let us denote $\widetilde{G}_{n,k}$ the *oriented* Grassmann manifold of *oriented* k–dimensional vector subspaces in $\mathbb{R}^n$, the space $SO(n)/(SO(k) \times SO(n-k))$.

We may suppose that $k \leq n-k$ for both of them.

Introduction

Cohomology ring of $G_{n,k}$

The cohomology ring of the Grassmann manifold $G_{n,k}$ is

$$H^*(G_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/I_{n,k},$$

where $\dim(w_i) = i$ and the ideal $I_{n,k}$ is generated by k homogeneous polynomials $\bar{w}_{n-k+1}, \bar{w}_{n-k+2}, \ldots, \bar{w}_n$, where each $\bar{w}_i$ denotes the i–dimensional component of the formal power series

$$1+(w_1+w_2+\cdots+w_k)+(w_1+w_2+\cdots+w_k)^2+(w_1+w_2+\cdots+w_k)^3+\cdots .$$

Each indeterminate w_i is a representative of the ith Stiefel-Whitney class $w_i(\gamma_{n,k})$ of the canonical k–plane bundle $\gamma_{n,k}$ over $G_{n,k}$.

Characteristic rank

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The notion of characteristic rank quantifies the degree up to which the cohomology ring is generated by Stiefel-Whitney classes.

Definition

Let X be a connected, finite CW–complex and ξ a vector bundle over X. The *characteristic rank* of the vector bundle ξ, denoted $\mathrm{charrank}(\xi)$, is the greatest integer q, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X;\mathbb{Z}_2)$ for $0 \leq j \leq q$ can be expressed as a polynomial in the Stiefel–Whitney classes $w_i(\xi)$ of ξ.

Cup-length

Let us recall the definition of the cup-length of a space as it turns out to be related to the characteristic rank.

Definition

For a path connected space X, we may define its $\mathbb{Z}_2$–cup-length $\mathrm{cup}_{\mathbb{Z}_2}(X)$ as the greatest number r such that there exist cohomology classes $x_1, \ldots, x_r \in H^*(X; \mathbb{Z}_2)$ in positive dimensions such that $x_1 \cdots x_r \neq 0$.

For the oriented Grassmann manifold $\widetilde{G}_{n,k}$ and its canonical bundle $\widetilde{\gamma}_{n,k}$ we have

$$\mathrm{cup}(\widetilde{G}_{n,k}) \leq 1 + \frac{k(n-k) - \mathrm{charrank}(\widetilde{\gamma}_{n,k}) - 1}{2}.$$

Characteristic rank of $\widetilde{\gamma}_{n,k}$

There is a covering projection $p\colon \widetilde{G}_{n,k} \to G_{n,k}$, which is universal for $(n,k) \neq (2,1)$. To this 2–fold covering, there is an associated line bundle ξ over $G_{n,k}$, such that $w_1(\xi) = w_1(\gamma_{n,k})$, to which we have Gysin exact sequence

$$\xrightarrow{\psi} H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\widetilde{G}_{n,k}) \xrightarrow{\psi} H^j(G_{n,k}) \xrightarrow{w_1} H^{j+1}(G_{n,k}) \rightarrow$$

where $H^j(G_{n,k}) \xrightarrow{w_1} H^{j+1}(G_{n,k})$ is the homomorphism given by the cup product with the first Stiefel–Whitney class $w_1(\xi) = w_1(\gamma_{n,k})$.

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Since $p^*\gamma_{n,k} \cong \widetilde{\gamma}_{n,k}$, the image $\mathrm{Im}(p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k}))$ is a subspace of the $\mathbb{Z}_2$–vector space $H^j(\widetilde{G}_{n,k})$ consisting only of cohomology classes, which can be expressed as polynomials in the Stiefel–Whitney characteristic classes of $\widetilde{\gamma}_{n,k}$.

Characteristic rank of $\widetilde{\gamma}_{n,k}$

This implies that the characteristic rank of $\widetilde{\gamma}_{n,k}$ is equal to the greatest integer q, such that the homomorphism $p^*\colon H^j(G_{n,k}) \to H^j(\widetilde{G}_{n,k})$ is surjective for all j, $0 \leq j \leq q$, or equivalently, that the homomorphism $w_1\colon H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ is injective for all j, $0 \leq j \leq q$.

Homomorphism $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$

Recall that

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, w_2, \ldots, w_k]/(\bar{w}_{n-k+1}, \ldots, \bar{w}_n).$$

Since in the quotient ring there are no relations in dimensions $n-k$ or lower, we always have $\mathrm{charrank}(\widetilde{\gamma}_{n,k}) \geq n-k-1$.

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Let $g_i \in \mathbb{Z}_2[w_2, \ldots, w_k]$ be the reduction of the polynomial $\bar{w}_i$ modulo w_1.
Denote $g_i(\gamma_{n,k})$ the corresponding cohomology class in $H^i(G_{n,k})$.

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For $i \in \{n-k+1, n-k+2, \ldots, n\}$ the cohomology classes $g_i(\gamma_{n,k})$ lie in the image of $w_1 : H^{i-1}(G_{n,k}) \longrightarrow H^i(G_{n,k})$.

Homomorphism $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$

By estimating the dimension of the image of the homomorphism $w_1 : H^j(G_{n,k}) \longrightarrow H^{j+1}(G_{n,k})$ we obtain the following proposition.[1]

Proposition

For a non-negative integer x, we associate with $H^{n-k+x+1}(G_{n,k})$ $(2 \leq k \leq n-k)$ the set

$$N_x(G_{n,k}) := \bigcup_{i=0}^{k-1} \{w_2^{b_2} \cdots w_k^{b_k} g_{n-k+1+i}; 2b_2 + 3b_3 + \cdots + kb_k = x - i\}.$$

If $x \leq n-k-1$ and the set $N_x(G_{n,k})$ is linearly independent, then

$$w_1 \colon H^{n-k+x}(G_{n,k}) \longrightarrow H^{n-k+x+1}(G_{n,k})$$

is a monomorphism.

[1] J. Korbaš, T. Rusin: *On the cohomology of oriented Grassmann manifolds.* Homology, Homotopy Appl., vol.18(2), 2016, 71–84.

The polynomials g_i

The sequence of polynomials g_i associated with $G_{n,k}$ is uniquely determined by k and it satisfies the recurrence formula

Recurrence formula

$g_i = w_2 g_{i-2} + w_3 g_{i-3} + \cdots + w_k g_{i-k}$.

Every $g_i \in \mathbb{Z}_2[w_2, \ldots, w_k]$ is a weighted homogeneous polynomial corresponding to a class in $H^i(G_{n,k})$. For any such polynomial we will simply say that it *lies in the dimension* i.

Each element of the set $N_x(G_{n,k})$ represents a polynomial lying in the dimension $n - k + x + 1$.

Linear independence of $N_x(G_{n,k})$

Any linear combination of elements from $N_x(G_{n,k})$ can be written as

$$f_x g_{n-k+1} + \cdots + f_{x-k+1} g_n, \tag{1}$$

where every $f_j \in \mathbb{Z}_2[w_2, \ldots, w_k]$ lies in dimension j.

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where every $f_j \in \mathbb{Z}_2[w_2, \ldots, w_k]$ lies in dimension j.
We can express this sum using products of formal power series. Consider an element g in the ring of formal power series $\mathbb{Z}_2[[w_2, \ldots, w_k]]$ given by

$$g = \sum_{i=0}^{\infty} g_i = 1 + g_1 + g_2 + \cdots .$$

For any $f \in \mathbb{Z}_2[w_2, \ldots, w_k]$ denote $[f \cdot g]_i$ the sum of all terms of formal power series $f \cdot g$ lying in dimension i.
The linear combination (1) is now exactly $[f \cdot g]_{n-k+x+1}$ for $f = f_x + \cdots + f_{x-k+1}$.

Linear independence of $N_x(G_{n,k})$

Thus linear independence of the set $N_x(G_{n,k})$ is equivalent to the condition that for any nonzero polynomial $f = f_x + \cdots + f_{x-k+1}$ we have $[f \cdot g]_{n-k+x+1} \neq 0$.

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To progress further we need to allow for polynomials that are not weighted homogeneous.

Definition

Let $i \in \mathbb{N}$ and $f \in \mathbb{Z}_2[w_2, \ldots, w_k]$ be a polynomial that is a sum of polynomials each of which lies in dimension $j \leq i$. We will say that such a polynomial f is *contained in dimension* i.

Linear independence of $N_x(G_{n,k})$

In this setting the recurrence relation for g_i translates to the following.

Definition

Suppose $f \in \mathbb{Z}_2[w_2, \ldots, w_k]$. Let us denote $\bar{f}$ the polynomial obtained from f by replacing each instance of w_k with the sum $1 + w_2 + \cdots + w_{k-1}$.

Lemma

Suppose $f \in \mathbb{Z}_2[w_2, \ldots, w_k]$. If f is of the form $f = f_x + f_{x-1} + \cdots f_{x-k+1}$, where every f_j is lying in dimension j, then $f \neq 0$ implies $\bar{f} \neq 0$.
If f is contained in dimension i, then $\bar{f}$ is as well, and $[f \cdot g]_i = [\bar{f} \cdot g]_i$.

Hence, to prove linear independence of $N_x(G_{n,k})$, it is sufficient to show that for any nonzero polynomial in indeterminates $w_2, \ldots, w_{k-1}$ contained in dimension x we have $[\bar{f} \cdot g]_{n-k+x+1} \neq 0$.

Linear independence of $N_x(G_{n,k})$

This adjustment impels us to view every g_i as a polynomial in one indeterminate w_k with coefficients in the ring $\mathbb{Z}_2[w_2, \ldots, w_{k-1}]$ and to consider $[\bar{f} \cdot g]_{n-k+x+1}$ as a linear combination of such polynomials.

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Example

Consider $N_4(G_{18,3}) = \left\{w_2^2 g_{16}, w_3 g_{17}, w_2 g_{18}\right\}$.
A linear combination of elements of this set is of the form
$\alpha_4 w_2^2 g_{16} + \alpha_3 w_3 g_{17} + \alpha_2 w_2 g_{18} = [f \cdot g]_{20}$ for $f = \alpha_4 w_2^2 + \alpha_3 w_3 + \alpha_2 w_2$.

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Instead of working with f, we take an arbitrary nonzero polynomial from $\mathbb{Z}_2[w_2]$ contained in dimension 4, say $\beta_4 w_2^2 + \beta_2 w_2 + \beta_0$.

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We just need to show that $[(\beta_4 w_2^2 + \beta_2 w_2 + \beta_0) \cdot g]_{20} \neq 0$.

Linear independence of $N_x(G_{n,k})$

Example

Now observe

$$\begin{aligned} g_{16} &= w_2^8 + \quad + w_2^5 w_3^2 + \cdots \\ g_{18} &= w_2^9 + \quad + w_2^3 w_3^4 + \cdots \\ g_{20} &= w_2^{10} + \quad + w_2 w_3^6 + \cdots \end{aligned}$$

and it should be clear that $[(\beta_4 w_2^2 + \beta_2 w_2 + \beta_0) \cdot g]_{20} \neq 0$, unless $\beta_4 = \beta_2 = \beta_0 = 0$.

Conclusion

As the example illustrated, the main idea is to find intervals for i such that g_i viewed as polynomials of w_k are in this "echelon" form and then deriving the corresponding conditions for n, k, x. One such result is the following.

Theorem

Let $k \geq 3$ and $t \geq \max\{3, \log_2(k-1)\}$. For any $x \geq 0$ and n such that

$$(k-1) \cdot 2^{t-1} + \frac{k-3}{k-1} \cdot 2^{t-1} + \frac{x}{k-1} - 1 < n \leq k \cdot 2^{t-1} - 1 - x$$

we have

$$\mathrm{charrank}(\widetilde{\gamma}_{n,k}) \geq n - k + x.$$

Conclusion

For any $k \geq 3$ and $t \geq \max\{3, \log_2(k-1)\}$ let us denote $x_{k,t}$ the smallest positive integer such that $2^t - x_{k,t}$ is divisible by $k-1$. We define an integer

$$n_{k,t} = k \cdot 2^{t-1} - \frac{2^t - x_{k,t}}{k-1} - 1.$$

Theorem

For any $k \geq 3$, $t \geq \max\{3, \log_2(k-1)\}$ and $a \geq 0$, such that a satisfies $ka + x_{k,t} - 1 \leq \frac{2^t - x_{k,t}}{k-1}$ we have

$$\operatorname{cup}(\widetilde{G}_{n_{k,t}+a,k}) \leq 1 + \frac{k(n_{k,t}-k) - (n_{k,t} - k + x_{k,t})}{2}.$$

Conclusion

Thank you.