

9. BUCHBERGER'S ALGORITHM.

55. Using Buchberger's algorithm, find a Gröbner basis of the ideal $(z - x^5, y - x^3) \subset k[x, y, z]$. Use

- (a) lexicographic ordering ($x > y > z$),
- (b) graded reverse lexicographic ordering.

Give also computations and reductions of S-polynomials.

56. Buchberger's algorithm is a generalization of more special cases which you already know. Try to recognize them:

- (a) Let $I = (f, g) \subset k[x]$ be an ideal in a polynomial ring with one variable. What can you say about Buchberger's algorithm in this case? (Try an example: for $I = (x^3 - 1, x^5 - 1) \subset \mathbb{R}[x]$, check (using Buchberger's algorithm) whether $x^3 + x^2 - 2$ belongs to I .)
- (b) Let $I \subset k[x_1, \dots, x_n]$ be an ideal generated by linear polynomials (and so defining a linear variety). What can you say about Buchberger's algorithm?

57. Let

$$g_1 = y^3 + 3y^2z^3, \quad g_2 = xyz + 3xz^4, \quad g_3 = 4x^2z - 7y^2$$

be polynomials in $k[x, y, z]$ and let $I = (g_1, g_2, g_3)$ be an ideal. Consider the lexicographic ordering of monomials with $x > y > z$. Is it possible to find $g \in I$ such that

$$\text{LT}(g) \notin (\text{LT}(g_1), \text{LT}(g_2), \text{LT}(g_3))?$$

If yes, find such a polynomial. If no, prove.

10. ELIMINATION USING GRÖBNER BASES

58. Using Gröbner bases, solve:

$$\begin{aligned} x^2 + y^2 + z^2 &= 4, \\ x^2 + 2y^2 &= 5, \\ xz &= 1. \end{aligned}$$

Help yourself with a computer algebra system. For example, in the lecture we briefly introduced Sage (<https://sagecell.sagemath.org/>), I recommend also Singular (<https://www.singular.uni-kl.de:8003/>).

59. Consider the ideal

$$I = (x^2 + y^2 + z^2 + 2, 3x^2 + 4y^2 + 4z^2 + 5) \subset k[x, y, z].$$

Let

$$I_1 = I \cap k[y, z].$$

be the first elimination ideal.

- (i) Compute I_1 .
- (ii) Check, that in case $k = \mathbb{C}$ it holds $V(I_1) = \pi_1(V(I))$, where π_1 is the projection $(a, b, c) \mapsto (b, c)$.
- (iii) Check, that in case $k = \mathbb{R}$ it holds $V(I) = \emptyset$, although the set $V(I_1)$ is infinite.