

7. RATIONAL FUNCTIONS AND RATIONAL MAPS.

45. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a regular matrix. Show that the rational map

$$f : \mathbb{A}^1 \rightarrow \mathbb{A}^1, \quad x \mapsto \frac{ax + b}{cx + d}$$

is dominant. Is f a birational equivalence?

46. Let X be an irreducible variety and let $f : X \rightarrow Y$ be a rational map. During the lecture we have mentioned (in the proof of Theorem 3.12), that f is dominant if and only if the pullback $f^* : k[Y] \hookrightarrow k(X)$ is an embedding.

- (a) Fill in the details of the proof: you should write out the proof of the implication “if f^* is an embedding, then f is dominant”.
- (b) Using (a), prove: If X is an irreducible variety and $f : X \rightarrow Y$ is a dominant map, then Y is also an irreducible variety.

47. Let $X = V(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3(\mathbb{C})$. Decompose X into irreducible components. Prove that each component is rational (i.e. birational to $\mathbb{A}^n(\mathbb{C})$ for some n).

48. The cubic curve $C = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2(\mathbb{C})$ is singular in $(0, 0)$. Sketch the curve (its real points). Find a birational equivalence of the curve with an affine line. (Hint: you can proceed like in the case of stereographic projection of a circle. In this case, consider lines through the singular point and project the curve onto the line $x = 1$.)

49. Let C be the cubic from the previous example, $C = V(y^2 - x^3 - x^2) \subset \mathbb{A}^2(\mathbb{C})$. Show that the function y/x is not regular, i.e. $y/x \notin \mathbb{C}[C]$. In which points of C is the function regular?

50. Find a birational equivalence of the varieties

$$\begin{aligned} X &= V(y^2 - x^3) \subset \mathbb{A}^2, \\ Y &= V(y^3 - x^4) \subset \mathbb{A}^2. \end{aligned}$$