

3. IDEALS AND VARIETIES, HILBERT'S BASIS THEOREM

18. Given two systems of linear equations decide whether they describe the same linear variety in $\mathbb{A}^3(\mathbb{R})$, i.e. find out whether $V(f_1, f_2, f_3) = V(g_1, g_2)$:

- (a) $f_1 = x + y + z - 1, f_2 = x - y + 2z - 4,$
 $g_1 = x + 5y - z + 5, g_2 = 3x + y + z - 2.$
- (b) $f_1 = 2x + 3y - z, f_2 = x + y - 1, f_3 = x + z - 3,$
 $g_1 = x + 3y - 2z + 3, g_2 = y - z + 2.$
- (c) Given are two linear varieties $X_1 = V(f_1, \dots, f_r), X_2 = V(g_1, \dots, g_s) \subset \mathbb{A}^n$
 $(f_i, g_j \in k[x_1, \dots, x_n] \text{ are linear}).$ Try to suggest an algorithm deciding whether $X_1 = X_2$.

19. In Example 1.3 (iv) in the lecture we described the two points set $X = \{(1, 2), (3, 4)\} \subset \mathbb{A}^2(\mathbb{Q})$ by polynomials in two ways. Show that

$$((x-1)(x-3), (x-1)(y-4), (y-2)(x-3), (y-2)(y-4)) = ((x-1)(x-3), x-y+1).$$

In other words, show that the two sets of polynomials generate the same ideal. (This claim is stronger than the claim that they describe the same algebraic variety.)

20. Let $X \subset \mathbb{A}^n(k)$ be a finite set. We know already that X is an affine variety. Show that X is the zero set of n polynomials, i.e. that there exist $f_1, \dots, f_n \in k[x_1, \dots, x_n]$ such that $X = V(f_1, \dots, f_n)$. (Hint: interpolation.)

21. Let $I, J \subset k[x_1, \dots, x_n]$ be ideals and let $X = V(I), Y = V(J)$ be corresponding algebraic varieties. What can you say about varieties $V(IJ), V(I \cap J), V(I + J)$?

22. Let k be a field. Is the ring $k[x_1, x_2, \dots]$ of polynomials in infinitely many variables noetherian?

23. Consider the set of real functions that are continuous on the interval $\langle 0, 1 \rangle \subset \mathbb{R}$. This set is a ring. Show that this ring is not noetherian.

24. Let $S \subset \mathbb{A}^n(k)$ and let $I(S) \subset k[x_1, \dots, x_n]$ (a set defined in the lecture). Show that $I(S)$ is an ideal in $k[x_1, \dots, x_n]$.

25. Prove Proposition 4.15 from the lecture. (You prove (i) and (ii) by just rewriting the definitions. The claim (iii) will follow from (i) and (ii).)