

1. THE NOTION OF ALGEBRAIC VARIETY

1. Sketch the following plane curves (algebraic varieties in $\mathbb{A}^2(\mathbb{R})$):

- $V(x^3 - y^2)$,
- $V(x^3 + x^2 - y^2)$,
- $V(x^4 - x^2 + y^2)$,
- $V((x^2 + y^2)^3 - 4x^2y^2)$,

It might be helpful to look for intersections of the curve with lines through $(0, 0)$. Try NOT to use any computer algebra system (matlab, maple,...).

2. Describe the algebraic variety $V(xy, xz, yz) \subset \mathbb{A}^3(\mathbb{R})$.

3. Show that the unit circle in $\mathbb{A}^2(\mathbb{R})$ together with its center is an algebraic variety, i.e. find a polynomial equation or a system of polynomial equations such that its/their solution is exactly the given set.

4. Show that the set

$$\{(a, b) \mid a + b \in \mathbb{Z}\}$$

is not an algebraic variety in $\mathbb{A}^2(\mathbb{R})$.

5. In $\mathbb{A}^2(\mathbb{R})$ consider the sets

$$M_1 = \{(1, 1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}), \dots, (\frac{1}{n}, \frac{1}{n})\}$$

$$M_2 = \{(1, 1), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3}), \dots, (\frac{1}{n}, \frac{1}{n}), \dots\}$$

For each of them find out whether it is an algebraic variety. If yes, find the defining polynomials. If not, prove.

6. Consider the set consisting of the point $(1, 1, 1)$, the point $(-1, -1, -1)$ and the points $\{(0, t, 0) \mid t \in \mathbb{C}\}$ (y -axis). Decide whether the given set is an algebraic variety in $\mathbb{A}^3(\mathbb{C})$. If yes, find the defining polynomials. If not, prove.

7. In $\mathbb{A}^3$ consider the three circles: each has its center in the beginning of the coordinates and radius 1, the first one lays in the xy -plane, the second one in the yz -plane and the third one in the xz -plane. Decide whether the three circles form an algebraic variety. If yes, find the defining polynomials. If not, prove.

8. Consider the set A of all 3×3 matrices over $\mathbb{R}$, so

$$A = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, a_{ij} \in \mathbb{R} \right\}.$$

This set can be naturally seen as the affine space $\mathbb{A}^9(\mathbb{R})$, where the coordinates are $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$.

- (a) Show that the set of all singular matrices in A is an algebraic variety in this space.
- (b) Show that the set of all matrices in A with the rank at most 1, is an algebraic variety in this space.

9. Consider the surface

$$V(3x^2 + 3y^2 + 3z^2 - 10xyz - 3)$$

in $\mathbb{A}^3(\mathbb{R})$. Find out whether the surface contains the line of intersection of the two planes

$$V(x - y - 2z + 2) \quad \text{and} \quad V(2x - 2y + z + 1).$$

10. Let $M \subset \mathbb{A}^1(\mathbb{C})$ consists of all points of complex affine line with the real coordinate. Is M an algebraic variety in $\mathbb{A}^1(\mathbb{C})$? Give reasons for your answer.