General rotations in 3D
Quaternions

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Computer Graphics (1)
Lecture 3

Spherical coordinates in 3D

- `Input:` a point $P \in \mathbb{E}^3(\mathbb{R})$
- `Output:` cartesian and spherical coordinates of P

- one point, multiple representations:
  - spherical:

$$r = |OP| \quad \rightsquigarrow r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \angle \mathbf{e}_1, \overline{OP*} \quad \rightsquigarrow \theta = \arccos \frac{x}{\sqrt{x^2 + y^2}} = $$

$$= \arcsin \frac{y}{\sqrt{x^2 + y^2}}$$

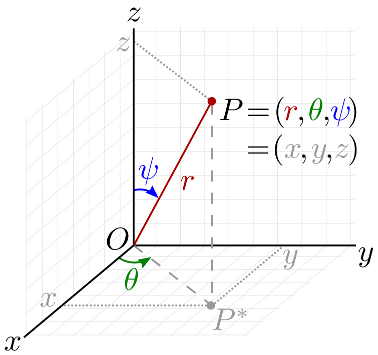
$$\psi = \angle \mathbf{e}_3, \overline{OP} \quad \rightsquigarrow \psi = \arccos \frac{z}{r}$$

  - cartesian:

$$x = r \cdot \cos\theta \cdot \sin\psi$$

$$y = r \cdot \sin\theta \cdot \sin\psi$$

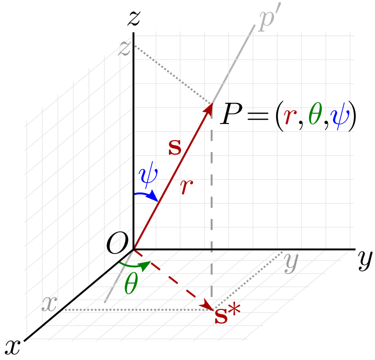
$$z = r \cdot \cos\psi$$



Original figure from: `https://en.wikipedia.org/wiki/Spherical_coordinate_system`

Rotation in 3D w.r.t. a general line

- Input: a line $p = \langle P, \mathbf{s} \rangle$, an angle φ, an object X (e.g. a point)
- Output: image of X under rotation around p



Original figure from: https://en.wikipedia.org/wiki/Spherical_coordinate_system

- rotation $\leftrightsquigarrow$ a composite transformation:
  - I. identify p with an axis (e.g. $\vec{z}$)
  - II. perform the rotation $\mathbf{R}_z(\varphi)$
  - III. de-identify p

  How:
  - I. identification $p \mapsto \vec{z}$:
    - a translation $P \mapsto P' \rightsquigarrow p \mapsto p' \quad (P' \equiv O)$
    - b rotation $\mathbf{R}_z(-\theta)$
    - c rotation $\mathbf{R}_y(-\psi)$
  - II. rotation $\mathbf{R}_z(\varphi)$
  - III. de-identification $\vec{z} \mapsto p$:
    - a rotation $\mathbf{R}_y(\psi)$ - inverse transf. to I.(c)
    - b rotation $\mathbf{R}_z(\theta)$ - inverse transf. to I.(b)
    - c translation $P' \mapsto P \rightsquigarrow p' \mapsto p$ - inverse transf. to I.(a)

- Result: multiplication of 7 matrices 4x4

$$\mathbf{X}' = \underbrace{\mathbf{T}_{P-O} \cdot \mathbf{R}_z(\theta) \cdot \mathbf{R}_y(\psi)}_{\text{III} = \text{I.}^{-1}} \cdot \underbrace{\mathbf{R}_z(\varphi)}_{\text{II.}} \cdot \underbrace{\mathbf{R}_y(-\psi) \cdot \mathbf{R}_z(-\theta) \cdot \mathbf{T}_{O-P}}_{\text{I.}} \cdot \mathbf{X}$$

Quaternions – definition

- $\mathbb{C} \sim \mathbb{R}^2$:

$$z = (a, b) \in \mathbb{R}^2 \rightsquigarrow a \cdot 1 + b \cdot i \qquad \text{with } i^2 = -1$$
$$z = |z| \cdot (\cos\varphi + i \sin\varphi) = |z| \cdot e^{i\varphi}$$

- $\mathbb{H} \sim \mathbb{C}^2$:

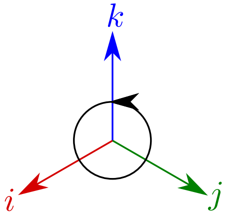
$$\begin{aligned} \mathbf{q} = (a + b \cdot i, c + d \cdot i) \in \mathbb{C}^2 \rightsquigarrow\ & (a + b \cdot i) \cdot 1 + (c + d \cdot i) \cdot j = \\ & a + b \cdot i + c \cdot j + d \cdot i \cdot j =: \\ & a + b \cdot i + c \cdot j + d \cdot k \\ & \text{with } i^2 = j^2 = k^2 = i \cdot j \cdot k = -1 \end{aligned}$$

$$\mathbf{q} = |\mathbf{q}| \cdot (\cos\varphi + \vec{u} \sin\varphi) = |\mathbf{q}| \cdot e^{\vec{u}\varphi}$$

where $\vec{u} = (u_1, u_2, u_3) = u_1 i + u_2 j + u_3 k$ is pure-imaginary unit quaternion

- multiplication of the imaginary units is non-commutative:

$$i \cdot j = k = -j \cdot i,$$
$$j \cdot k = i = -k \cdot j,$$
$$k \cdot i = j = -i \cdot k,$$



Quaternions – notation and operations

- notation:

$$a + b \cdot i + c \cdot j + d \cdot k \rightsquigarrow (a, b, c, d) =: (q_0, q_1, q_2, q_3) =: \mathbf{q}$$

- addition:

$$\mathbf{q} + \mathbf{q}' = (q_0 + q_0', q_1 + q_1', q_2 + q_2', q_3 + q_3')$$

- multiplication by $r \in \mathbb{R}$:

$$r \cdot \mathbf{q} = (r \cdot q_0, r \cdot q_1, r \cdot q_2, r \cdot q_3)$$

- multiplication by a quaternion:

$$\begin{aligned}
\mathbf{q} \cdot \mathbf{q}' &= (q_0 + q_1 i + q_2 j + q_3 k) \cdot (q_0' + q_1' i + q_2' j + q_3' k) = \\
&= (q_0 \cdot q_0' - q_1 \cdot q_1' - q_2 \cdot q_2' - q_3 \cdot q_3') \cdot 1 + \\
&\quad + (q_0 \cdot q_1' + q_1 \cdot q_0' + q_2 \cdot q_3' - q_3 \cdot q_2') \cdot i + \\
&\quad + (q_0 \cdot q_2' + q_2 \cdot q_0' + q_3 \cdot q_1' - q_1 \cdot q_3') \cdot j + \\
&\quad + (q_0 \cdot q_3' + q_3 \cdot q_0' + q_1 \cdot q_2' - q_2 \cdot q_1') \cdot k
\end{aligned}$$

Quaternions – properties

- conjugation:

$$\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k \quad \rightsquigarrow \quad \mathbf{q} \ = q_0 + \vec{q}$$
$$\mathbf{q}^* := q_0 - q_1 i - q_2 j - q_3 k \quad \rightsquigarrow \quad \mathbf{q}^* = q_0 - \vec{q}$$

where $\vec{q} = (q_1, q_2, q_3)$

- properties of conjugated quaternions:

$$(\mathbf{q}^*)^* = \mathbf{q},$$
$$\mathbf{q} \cdot \mathbf{q}^* = \mathbf{q}^* \cdot \mathbf{q},$$
$$(\mathbf{p} \cdot \mathbf{q})^* = \mathbf{q}^* \cdot \mathbf{p}^*,$$

If $q_0 = 0$, then $\mathbf{q}$ is pure-imaginary

- length:

$$\|\mathbf{q}\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = \sqrt{\mathbf{q} \cdot \mathbf{q}^*} = \sqrt{\mathbf{q}^* \cdot \mathbf{q}}$$

- normalization $\leftrightsquigarrow$ unit quaternion:

$$\mathbf{q}_u := \frac{\mathbf{q}}{\|\mathbf{q}\|} = \frac{\mathbf{q}}{\sqrt{\mathbf{q} \cdot \mathbf{q}^*}}$$

- inverse (reciprocal) quaternion:

$$\mathbf{q}^{-1} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|^2} \leftrightsquigarrow \frac{1}{\mathbf{q}} = \frac{\mathbf{q}^*}{\mathbf{q} \cdot \mathbf{q}^*} = \frac{\mathbf{q}^*}{(\sqrt{\mathbf{q} \cdot \mathbf{q}^*})^2} = \frac{\mathbf{q}^*}{\|\mathbf{q}\|^2}$$

Quaternions

- quaternions multiplication (another way):

$$\mathbf{pq} = (p_0, \vec{p}) \cdot (q_0, \vec{q}) = p_0 q_0 - \vec{p} \cdot \vec{q} + p_0 \vec{q} + q_0 \vec{p} + \vec{p} \times \vec{q}$$

  where $\vec{p} = (p_1, p_2, p_3)$ and $\vec{q} = (q_1, q_2, q_3)$

- quaternion rotation operator $L_q()$ on vector $\vec{v} \in \mathbb{R}^3$:

$$\begin{aligned} L_q(\vec{v}) &= \mathbf{q}\vec{v}\mathbf{q}^* = \dots \\ &= (q_0^2 - \|\vec{q}\|^2)\vec{v} + 2(\vec{q} \cdot \vec{v})\vec{q} + 2q_0(\vec{q} \times \vec{v}) \end{aligned}$$

  where $\mathbf{q}$ is quaternion, with $\|\vec{u}\| = 1$, which can be written in the form:

$$\mathbf{q} = q_0 + \vec{q} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\vec{u}$$

Quaternions – rotations in 3D

- Input: an object X (e.g. a point), a line $p = \langle P, \vec{u} \rangle$, an angle of rotation θ
- Output: image X' of X under rotation around p by θ
- Preprocessing:
  1. we assume $p \ni O(0,0,0)$ (translate P to O if necessary)
  2. set the object coordinates X as a vector $\vec{v}$:
  $$X = (x, y, z) \quad \rightsquigarrow \quad \vec{v} := X$$
  3. encode the axis of rotation (the line p) using quaternions:
     - normalize the direction vector $\vec{u}$ of p : $\quad \vec{u} := \frac{\vec{u}}{\|\vec{u}\|}$,
     - encode $\vec{u}$ as a pure-imaginary quaternion $\mathbf{q_u}$:
     $$\vec{u} = (u_x, u_y, u_z) \quad \rightsquigarrow \quad \mathbf{q_u} := u_x i + u_y j + u_z k$$
  4. encode the rotation as a quaternion and find its conjugation:
  $$\mathbf{q_{u\theta}} := \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{q_u} = q_0 + \vec{q}, \qquad \mathbf{q^*_{u\theta}} := \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\mathbf{q_u} = q_0 - \vec{q}$$
- Procedure:
  1. perform the rotation around p using the quaternions from Preprocessing:
  $$X' := \mathbf{q_{u\theta}} \vec{v} \mathbf{q^*_{u\theta}} = (q_0^2 - \|\vec{q}\|^2)\vec{v} + 2(\vec{q} \cdot \vec{v})\vec{q} + 2q_0(\vec{q} \times \vec{v})$$
  2. we get the image X' of X:
  $$X' = (x', y', z')$$

Quaternions – matrix representation of the 3D rotation

- The multiplication of quaternions is non-commutative:
  - $\rightsquigarrow$ problems when using computation on PC (a specialized library is needed)
  - $\rightsquigarrow$ another representation of the rotations
- `Input:` the point $X = (x, y, z)$, the quaternion encoding the rotation (see above):

$$\mathbf{q}_{\mathbf{u}\theta} := \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\mathbf{q}_{\mathbf{u}} =: (q_0, q_1, q_2, q_3)$$

- `Output:` the image X' of X under the rotation around p by θ
- `Preprocessing:` assemble the matrix

$$\mathbb{M}_{\mathbf{q}} := \begin{pmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_1q_2+q_0q_3) & 1-2(q_1^2+q_3^2) & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & 1-2(q_1^2+q_2^2) \end{pmatrix}$$

- `Procedure:`

$$\mathbf{X}' := \mathbb{M}_{\mathbf{q}} \cdot \mathbf{X}$$

$$(x', y', z')^\top = \mathbb{M}_{\mathbf{q}} \cdot (x, y, z)^\top \quad \rightsquigarrow \quad X' = (x', y', z')$$

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