

Transformations in 2D and 3D

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Computer Graphics (1)
lecture no. 1



- 1 Affine transformations
- 2 Affine transformations in 2D
- 3 Homogeneous coordinates, matrices of affine transformations in 2D
- 4 Affine transformations in 3D



Affine transformation

- A mapping

$$f: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is *affine* if it preserves:

- *collinearity*: the image of a line segment is either a line segment or a point
- *λ -ratio*: for collinear A, B, C and $\lambda \in \mathbb{R}$

$$A - C = \lambda(B - C) \implies f(A) - f(C) = \lambda(f(B) - f(C))$$

- The mapping

$$f: \mathbb{R}^m \rightarrow \mathbb{R}^n$$

$$(x_1, \dots, x_m) \mapsto (x'_1, \dots, x'_n)$$

is affine $\iff \exists c_{ij} \in \mathbb{R}$ such that

$$x'_1 = c_{11}x_1 + \dots + c_{1m}x_m + c_{10}$$

$$\vdots$$

$$x'_n = c_{n1}x_1 + \dots + c_{nm}x_m + c_{n0}$$

- $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *transformation* of $\mathbb{R}^n$ if it is invertible: $|c_{ij}|_{i,j=1}^n \neq 0$



Affine transformations in 2D

- The mapping

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

given by

$$x' = c_{11}x_1 + c_{12}y + c_{10}$$

$$y' = c_{21}x_1 + c_{22}y + c_{20}$$

such that

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0$$

- E.g. translation, reflection, scaling, shear, rotation (and their compositions)



Translation in 2D

- Input: vector $t = (t_x, t_y)$, object (e.g. point $X = (x, y)$)
- Output: image of X after translation
- transformation equations:

$$x' = x + t_x$$

$$y' = y + t_y$$

- inverse transformation:

$$x = x' - t_x$$

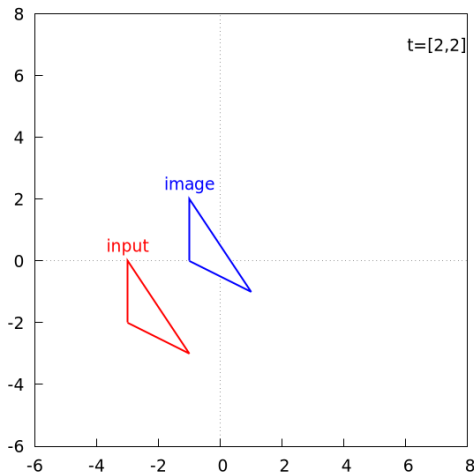
$$y = y' - t_y$$

- matrix form:

$$\mathbf{X}' = \mathbf{T} + \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} t_x \\ t_y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$

Triangle and its translation



Reflection in 2D

- Input: axis (an arbitrary line), object (e.g. point $X = (x, y)$)
- Output: image of X after reflection
- transformation equations:

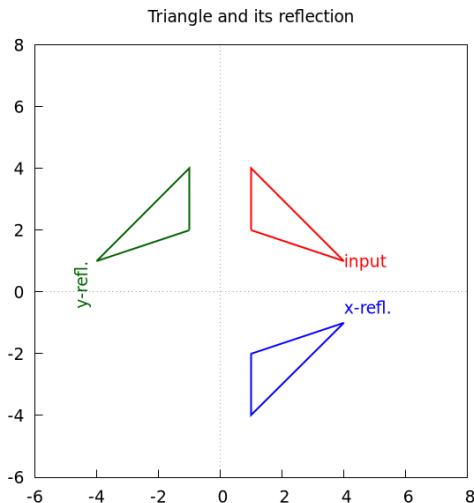
$$\begin{array}{lcl} x' = 1 \cdot x & & x' = (-1) \cdot x \\ y' = (-1) \cdot y & \text{or} & y' = 1 \cdot y \end{array}$$

- inverse transformation: reflection
- matrix form:

$$\mathbf{X}' = \mathbf{Z} \cdot \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} z_x & 0 \\ 0 & z_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

with $z_x, z_y \in \{1, -1\}$ as above



Scaling in 2D

- Input: scale factors $s_x, s_y \neq 0$, objekt (e.g. point $X = (x, y)$)
- Output: image of X after scaling
- transformation equations:

$$x' = s_x \cdot x$$

$$y' = s_y \cdot y$$

- inverse transformation:

$$x = \frac{1}{s_x} \cdot x'$$

$$y = \frac{1}{s_y} \cdot y'$$

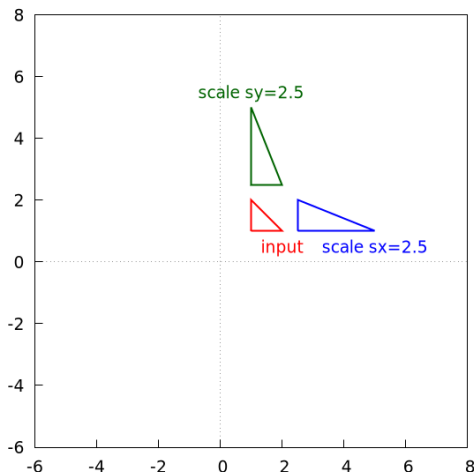
- matrix form:

$$\mathbf{x}' = \mathbf{S} \cdot \mathbf{x}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- x -reflection: $s_x = 1, s_y = -1$
- y -reflection: $s_x = -1, s_y = 1$

Triangle and its scaling



Shear in 2D

- Input: shear factors k_x, k_y , object (e.g. point $X = (x, y)$)
- Output: image of X after shearing
- transformation equations:

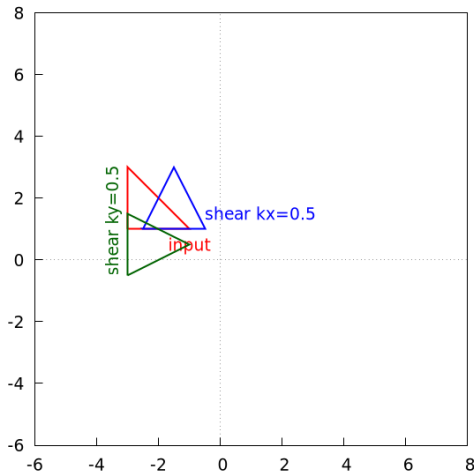
$$\begin{array}{lcl} x' = x + k_x y & \text{or} & x' = x \\ y' = y & & y' = y + k_y x \end{array}$$

- inverse transformation:
homework
- matrix form:

$$\mathbf{X}' = \mathbf{K} \cdot \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & k_x \\ k_y & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Triangle and its shearing



Rotation in 2D

- Input: angle φ , object (e.g. point $X = (x, y)$)
- Output: image of X after rotation
- transformation equations:

$$x' = x \cdot \cos \varphi - y \cdot \sin \varphi$$

$$y' = x \cdot \sin \varphi + y \cdot \cos \varphi$$

- inverse transformation:

$$x = x' \cdot \cos \varphi + y' \cdot \sin \varphi$$

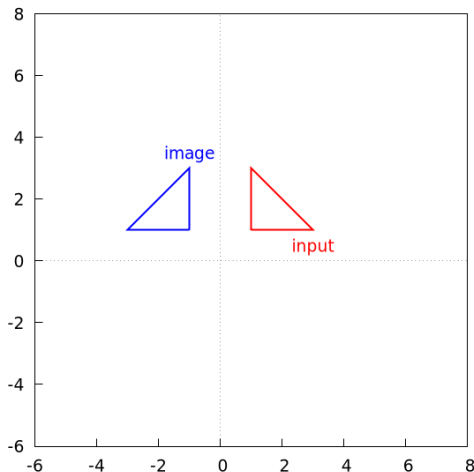
$$y = -x' \cdot \sin \varphi + y' \cdot \cos \varphi$$

- matrix form:

$$\mathbf{X}' = \mathbf{R} \cdot \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Triangle and its rotation



Homogeneous coordinates, matrices of affine transformations

- almost all transformations had the same matrix form:

$$\mathbf{X}' = \mathbf{M} \cdot \mathbf{X}$$

with $\mathbf{M}$ equal to $\mathbf{Z}, \mathbf{S}, \mathbf{K}, \mathbf{R}$; translation was the exception:

$$\mathbf{X}' = \mathbf{T} + \mathbf{X}$$

- ?? is there a unified form?
 - Why is it useful? Composition of transformations $\leftrightarrow$ matrix multiplication
 - How do we obtain it? Homogeneous coordinates:
 - point: $\mathbf{X} = (x \ y)^T \rightsquigarrow \tilde{\mathbf{X}} = (x \ y \ 1)^T$
 - vector: $\mathbf{u} = (u \ v)^T \rightsquigarrow \tilde{\mathbf{u}} = (u \ v \ 0)^T$
 - output: a 3×3 matrix $\mathbf{M} \rightsquigarrow$ we need 3 non-collinear points
- affine transformation matrix: $\mathbf{X}' = \mathbf{M} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



Matrices of affine transformations in 2D and their inverses

- translation: $\mathbf{X}' = \mathbf{T} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- scaling & reflection: $\mathbf{X}' = \mathbf{S} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- shear: $\mathbf{X}' = \mathbf{K} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & k_x & 0 \\ k_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- rotation: $\mathbf{X}' = \mathbf{R} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- affine transf. matrix: $\mathbf{X}' = \mathbf{M} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse translation: $\mathbf{T}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse scaling: $\mathbf{S}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse shear: $\mathbf{K}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} 1 & \frac{k_x}{k_x \cdot k_y - 1} & 0 \\ \frac{k_y}{k_x \cdot k_y - 1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse rotation: $\mathbf{R}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse affine transf.: $\mathbf{M}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$



Affine transformations in 3D

- The mapping

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

given by the equations

$$x' = c_{11}x + c_{12}y + c_{13}z + c_{10}$$

$$y' = c_{21}x + c_{22}y + c_{23}z + c_{20}$$

$$z' = c_{31}x + c_{32}y + c_{33}z + c_{30}$$

such that

$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \neq 0$$

- Translation, reflection, scaling, shear, rotation: (almost) as in 2D



Translation in 3D

- $\rightsquigarrow$ straightforward extension from 2D to 3D
- Input: vector $t = (t_x, t_y, t_z)$, object X
- Output: image of X after translation
- translation: $\mathbf{X}' = \mathbf{T} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- inverse translation: $\mathbf{T}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$



Reflection and scaling in 3D

- $\leadsto$ reflection in a coordinate plane $x = 0$, $y = 0$, or $z = 0$
- Input: reflection plane, object X
- Output: image of X after reflection
- reflection:

$$\mathbf{x}' = \mathbf{Z} \cdot \mathbf{x},$$

$$\mathbf{Z}^{-1} \cdot \mathbf{x}' = \mathbf{x}$$

$$\mathbf{Z}_x = \mathbf{Z}_x^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Z}_y = \mathbf{Z}_y^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Z}_z = \mathbf{Z}_z^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

- $\leadsto$ straightforward extension from 2D to 3D
- Input: $s_x, s_y, s_z \neq 0$, object X
- Output: image of X after scaling
- scaling:

$$\mathbf{x}' = \mathbf{S} \cdot \mathbf{x}$$

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- inverse scaling:

$$\mathbf{s}^{-1} \cdot \mathbf{x}' = \mathbf{x}$$

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Rotation about the x-axis

- Input: angle φ , object X
- Output: image of X after rotation about the x-axis

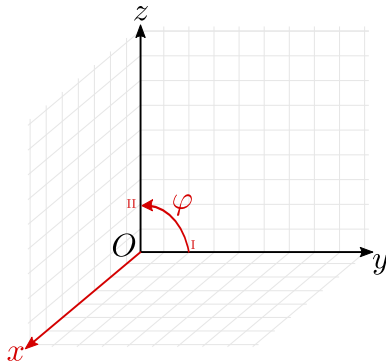
$$x' = x$$

$$\text{I: } y' = y \cdot \cos \varphi - z \cdot \sin \varphi$$

$$\text{II: } z' = y \cdot \sin \varphi + z \cdot \cos \varphi$$

$$R_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi & 0 \\ 0 & \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_x^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi & 0 \\ 0 & -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Spherical_coordinate_system



Rotation about the y -axis

- Input: angle φ , object X
- Output: image of X after rotation about the y -axis

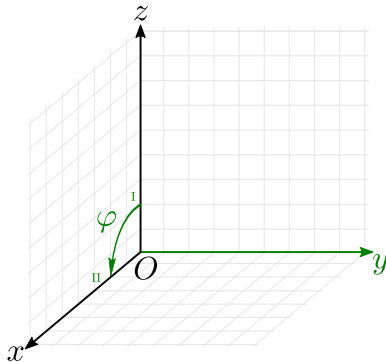
$$\text{II: } x' = x \cdot \cos \varphi + z \cdot \sin \varphi$$

$$y' = y$$

$$\text{I: } z' = -x \cdot \sin \varphi + z \cdot \cos \varphi$$

$$R_y = \begin{pmatrix} \cos \varphi & 0 & \sin \varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \varphi & 0 & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_y^{-1} = \begin{pmatrix} \cos \varphi & 0 & -\sin \varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin \varphi & 0 & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Spherical_coordinate_system



Rotation about the z-axis

- Input: angle φ , object X
- Output: image of X after rotation about the z-axis
- $\leadsto$ direct extension of the 2D rotation equations

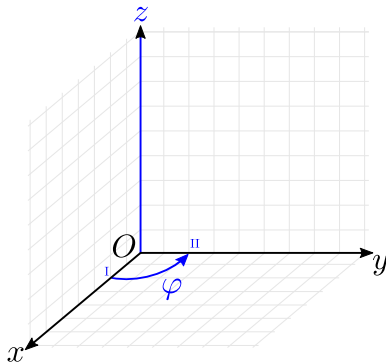
$$\text{I: } x' = x \cdot \cos \varphi - y \cdot \sin \varphi$$

$$\text{II: } y' = x \cdot \sin \varphi + y \cdot \cos \varphi$$

$$z' = z$$

$$R_z = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_z^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 & 0 \\ -\sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



https://en.wikipedia.org/wiki/Spherical_coordinate_system

