Transformations in 2D and 3D

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Computer Graphics (1)
Lecture 2

Affine transformation

- A map

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

  is *affine* if it preserves
  - *collinearity:* image of a line is a either a line or a point
  - the λ*-ratio:* for A, B, C collinear and $\lambda \in \mathbb{R}$

$$A - C = \lambda(B - C) \implies f(A) - f(C) = \lambda(f(B) - f(C))$$

- A map

$$f : \mathbb{R}^m \to \mathbb{R}^n$$
$$(x_1, \ldots, x_m) \mapsto (x'_1, \ldots, x'_n)$$

  is affine $\iff \exists c_{ij} \in \mathbb{R}$ s.t.

$$x'_1 = c_{11}x_1 + \ldots + c_{1m}x_m + c_{10}$$
$$\vdots$$
$$x'_n = c_{n1}x_1 + \ldots + c_{nm}x_m + c_{n0}$$

- $f : \mathbb{R}^n \to \mathbb{R}^n$ is a *transformation* of $\mathbb{R}^n$ if it is invertible: $|c_{ij}|_{i,j=1}^{n} \neq 0$

Affine transformation in 2D

- A map

$$f : \mathbb{R}^2 \to \mathbb{R}^2$$

given by

$$x' = c_{11}x_1 + c_{12}y + c_{10}$$
$$y' = c_{21}x_1 + c_{22}y + c_{20}$$

s.t.

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0$$

- E.g. a translation, a reflection, a scaling, a shearing, a rotation (and their compositions)

Translations in 2D

- `Input:` a vector $t = (t_x, t_y)$, an object (e.g. a point $X = (x, y)$)
- `Output:` image of X under translation
- transformation equations:

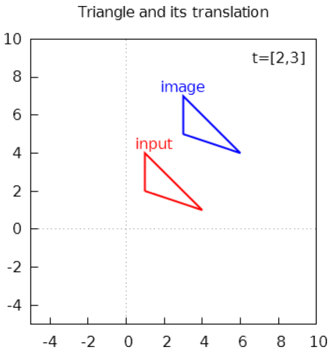
$$x' = x + t_x$$
$$y' = y + t_y$$

- inverse transformation:

$$x = x' - t_x$$
$$y = y' - t_y$$

- matrix form:

$$\mathbf{X}' = \mathbf{T} + \mathbf{X}$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} t_x \\ t_y \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix}$$



Reflections in 2D

- Input: an axis (in gen. any line), an object (e.g. a point $X = (x, y)$)
- Output: image of X under reflection
- transformation equations:

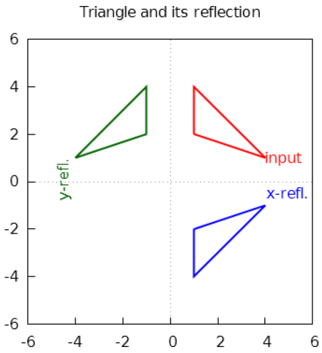
$$\begin{aligned} x' &= 1 \cdot x \\ y' &= (-1) \cdot y \end{aligned} \quad \text{resp.} \quad \begin{aligned} x' &= (-1) \cdot x \\ y' &= 1 \cdot y \end{aligned}$$

- inverse transformation: a reflection is its own inverse
- matrix form:

$$\mathbf{X}' = \mathbf{Z} \cdot \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} z_x & 0 \\ 0 & z_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

with $z_x, z_y \in \{1, -1\}$ chosen as above



Scaling in 2D

- `Input:` scaling factors s_x, s_y (non-zero), an object (e.g. a point $X = (x, y)$)
- `Output:` image of X under scaling
- transformation equations:

$$x' = s_x \cdot x$$
$$y' = s_y \cdot y$$

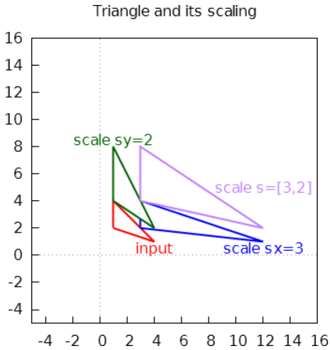
- inverse:

$$x = \frac{1}{s_x} \cdot x'$$
$$y = \frac{1}{s_y} \cdot y'$$

- matrix form:

$$\mathbf{X}' = \mathbf{S} \cdot \mathbf{X}$$
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- x-reflection: $s_x = 1, s_y = -1$
- y-reflection: $s_x = -1, s_y = 1$
- $\rightsquigarrow$ a reflection is a special (non-intuitive) case of scaling



Shearing in 2D

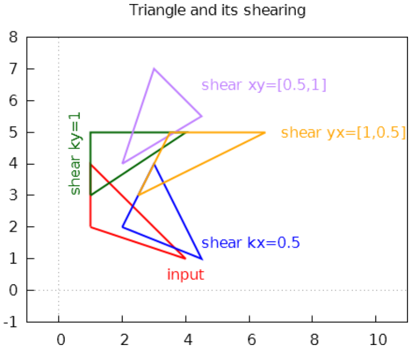
- `Input:` shearing factor k_x, k_y, object (e.g. a point $X = (x, y)$)
- `Output:` image of X under shearing
- transformation equations:

$$x' = x + k_x y \qquad\qquad x' = x$$
$$y' = y \qquad \text{resp.} \qquad y' = y + k_y x$$

- matrix form:

$$\mathbf{X}' = \mathbf{K} \cdot \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & k_x \\ k_y & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



Rotation in 2D

- `Input:` angle φ, object (e.g. a point $X = (x, y)$)
- `Output:` image of X under rotation
- transformation equations:

$$x' = x \cdot \cos\varphi - y \cdot \sin\varphi$$
$$y' = x \cdot \sin\varphi + y \cdot \cos\varphi$$

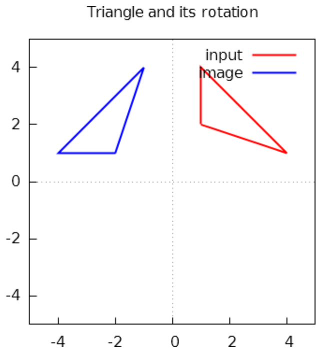
- inverse:

$$x = \quad x' \cdot \cos\varphi + y' \cdot \sin\varphi$$
$$y = -x' \cdot \sin\varphi + y' \cdot \cos\varphi$$

- matrix form:

$$\mathbf{X}' = \mathbf{R} \cdot \mathbf{X}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



Augmented coordinates. Matrix of an affine transformation.

- almost all the transformations have the same form:

$$\mathbf{X}' = \mathbf{M} \cdot \mathbf{X}$$

with $\mathbf{M}$ being $\mathbf{Z}, \mathbf{S}, \mathbf{K}, \mathbf{R}$, the only exception is the translation:

$$\mathbf{X}' = \mathbf{T} + \mathbf{X}$$

- ?? unified form:
  - WHY: composition of maps $\leftrightsquigarrow$ multiplication of matrices
  - HOW: augmented (homogeneous) coordinates:
  - point: $X = (x\ y)^\top \rightsquigarrow X^\sim = (x\ y\ 1)^\top$
  - vector: $\mathbf{u} = (u\ v)^\top \rightsquigarrow \mathbf{u}^\sim = (u\ v\ 0)^\top$
  - OUT: 3×3 matrix $\mathbf{M}$ $\rightsquigarrow$ we need 3 non-collinear points
- matrix of affine transformation: $\mathbf{X}' = \mathbf{M} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Matrices of affine transformations and their inverses

- translation: $\mathbf{X}' = \mathbf{T} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- scaling & reflection: $\mathbf{X}' = \mathbf{S} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- shearing: $\mathbf{X}' = \mathbf{K} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & k_x & 0 \\ k_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- rotation: $\mathbf{X}' = \mathbf{R} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- matrix of affine transfor.: $\mathbf{X}' = \mathbf{M} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse translation: $\mathbf{T}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse scaling: $\mathbf{S}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse of shearing: $\mathbf{K}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} 1 & \frac{k_x}{k_x \cdot k_y - 1} & 0 \\ \frac{k_y}{k_x \cdot k_y - 1} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse of rotation: $\mathbf{R}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- inverse of affine transf.: $\mathbf{M}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Affine transformation in 3D

- A map
$$f : \mathbb{R}^3 \to \mathbb{R}^3$$
given by
$$\begin{aligned} x' &= c_{11}x + c_{12}y + c_{13}z + c_{10} \\ y' &= c_{21}x + c_{22}y + c_{23}z + c_{20} \\ z' &= c_{31}x + c_{32}y + c_{33}z + c_{30} \end{aligned}$$
s.t.
$$\begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \neq 0$$
- Translation, a reflection, a scaling, a shearing, a rotation: (almost) as before

Translations in 3D

- $\rightsquigarrow$ a straightforward generalization of the planar case
- `Input:` vector $t = (t_x, t_y, t_z)$, object X
- `Output:` image of X under translation
- translation: $\mathbf{X}' = \mathbf{T} \cdot \mathbf{X}$

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

- inverse translation: $\mathbf{T}^{-1} \cdot \mathbf{X}' = \mathbf{X}$

$$\begin{pmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Reflections and scaling in 3D

- $\rightsquigarrow$ reflection w.r.t. a x- (y-, z-) plane
- `Input:` a reflection plane, object X
- `Output:` image of X under reflection
- reflection:

$$\mathbf{X}' = \mathbf{Z} \cdot \mathbf{X},$$
$$\mathbf{Z}^{-1} \cdot \mathbf{X}' = \mathbf{X}$$

$$\mathbf{Z}_x = \mathbf{Z}_x^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Z}_y = \mathbf{Z}_y^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{Z}_z = \mathbf{Z}_z^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- $\rightsquigarrow$ a straightforward generalization of the planar case
- `Input:` scaling factors s_x, s_y, s_z (non-zero), object X
- `Output:` image of X under scaling
- scaling:

$$\mathbf{X}' = \mathbf{S} \cdot \mathbf{X}$$

$$\mathbf{S} = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- inverse scaling:

$$\mathbf{S}^{-1} \cdot \mathbf{X}' = \mathbf{X}$$

$$\mathbf{S}^{-1} = \begin{pmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

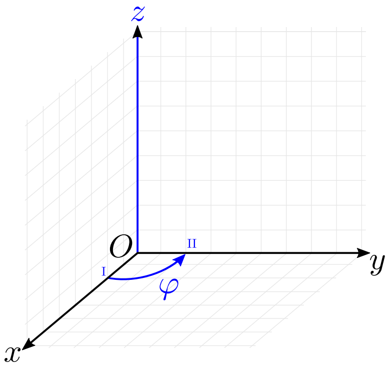
Rotation in 3D w.r.t. the z-coordinate axis

- `Input:` an angle φ, an object X
- `Output:` image of X under rotation around the z-axis
- $\rightsquigarrow$ a simple transition from plane to 3D

$$
\begin{aligned}
\text{I:}\quad & x' = x \cdot \cos\varphi - y \cdot \sin\varphi \\
\text{II:}\quad & y' = x \cdot \sin\varphi + y \cdot \cos\varphi \\
& z' = z
\end{aligned}
$$

$$
\mathbf{R}_z = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 & 0 \\ \sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
\mathbf{R}_z^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$



Original figure from: `https://en.wikipedia.org/wiki/Spherical_coordinate_system`

Rotation in 3D w.r.t. the x-coordinate axis

- `Input:` an angle φ, an object X
- `Output:` image of X under rotation around the x-axis

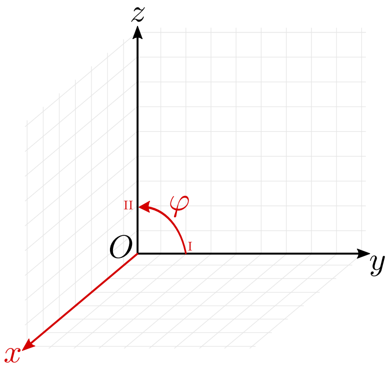
$$x' = x$$

$$\text{I:} \quad y' = y \cdot \cos\varphi - z \cdot \sin\varphi$$

$$\text{II:} \quad z' = y \cdot \sin\varphi + z \cdot \cos\varphi$$

$$\mathbf{R}_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_x^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Original figure from: `https://en.wikipedia.org/wiki/Spherical_coordinate_system`

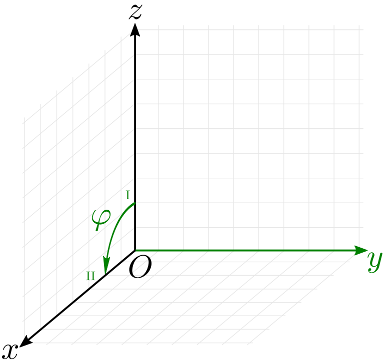
Rotation in 3D w.r.t. the y-coordinate axis

- `Input:` an angle φ, an object X
- `Output:` image of X under rotation around the y-axis

$$\begin{aligned} \text{II:} \quad x' &= x \cdot \cos\varphi + z \cdot \sin\varphi \\ y' &= y \\ \text{I:} \quad z' &= -x \cdot \sin\varphi + z \cdot \cos\varphi \end{aligned}$$

$$\mathbf{R}_y = \begin{pmatrix} \cos\varphi & 0 & \sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{R}_y^{-1} = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\varphi & 0 & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



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