

AFFINE TRANSFORMATIONS

1. INTRODUCTION

Definition 1.1. A mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called *affine* if

- it preserves collinearity, i.e. a line is mapped either to a line or to a single point,
- it preserves ratios along a line, i.e. if for collinear points A, B, C we have $C - A = \lambda(B - A)$ for some $\lambda \in \mathbb{R}$, then $f(C) - f(A) = \lambda(f(B) - f(A))$ as well.

An affine mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called an *affine transformation of $\mathbb{R}^n$* if it is invertible.

Proposition 1.2. A mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^m, (x_1, \dots, x_n) \mapsto (x'_1, \dots, x'_m)$ is affine if there exist constants $c_{ij} \in \mathbb{R}$ such that, for every point,

$$\begin{aligned} x'_1 &= c_{11}x_1 + \dots + c_{1n}x_n + c_{10} \\ x'_2 &= c_{21}x_1 + \dots + c_{2n}x_n + c_{20} \\ &\dots \\ x'_m &= c_{m1}x_1 + \dots + c_{mn}x_n + c_{m0}. \end{aligned}$$

For $m = n$, such a mapping is a transformation if the determinant $|c_{ij}|_{i,j=1}^n \neq 0$.

2. TRANSFORMATIONS IN $\mathbb{R}^2$

By the preceding proposition, every affine transformation can be described by the equations

$$\begin{aligned} x' &= c_{11}x + c_{12}y + c_{10} \\ y' &= c_{21}x + c_{22}y + c_{20}, \end{aligned} \quad \text{where} \quad \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0.$$

The real numbers c_{ij} uniquely determine this affine transformation.

2.1. Translation and scaling. *Translation* by the vector (t_x, t_y) is an affine transformation described by the linear equations

$$\begin{aligned} x' &= x + t_x \\ y' &= y + t_y. \end{aligned}$$

Scaling (along the coordinate axes) is described by the equations

$$\begin{aligned} x' &= s_x x \\ y' &= s_y y, \end{aligned} \quad \text{where } s_x, s_y \neq 0.$$

This is scaling about the center $(0, 0)$.

Example 2.1. Suppose we want to describe a scaling transformation about a center (a, b) that is not the origin. This transformation can be obtained by composing three familiar transformations:

- (1) $T_{(-a, -b)}$ – translation by $(-a, -b)$, moving the center of scaling to the origin,
- (2) S_{s_x, s_y} – scaling with the required scale factors s_x, s_y ,
- (3) $T_{(a, b)}$ – translation back.

Thus the resulting transformation is $T_{(a, b)} \circ S_{s_x, s_y} \circ T_{(-a, -b)}$. Its equations can be obtained by successively substituting the transformed coordinates into the individual equations; for implementation purposes, however, it is more convenient to express these mappings using matrices.

It is useful to employ homogeneous coordinates. A point with coordinates (a, b) and a vector with coordinates (u, v) will be represented, respectively, by the 3×1 matrices

$$\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}.$$

Then translation by the vector (t_x, t_y) can be written as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

and scaling takes the form

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Thus each of these transformations is uniquely represented by its 3×3 matrix. If, in the previous example, we denote the matrix of translation by $(-a, -b)$ by M_1 , the scaling matrix by M_2 , and the matrix of translation by (a, b) by M_3 , we obtain:

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} &= T_{(a,b)} \circ S_{s_x,s_y} \circ T_{(-a,-b)} \left(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right) = T_{(a,b)} (S_{s_x,s_y} (T_{(-a,-b)} \left(\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \right))) = \\ &= T_{(a,b)} (S_{s_x,s_y} (M_1 \begin{pmatrix} x \\ y \\ 1 \end{pmatrix})) = T_{(a,b)} (M_2 M_1 \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}) = M_3 M_2 M_1 \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}. \end{aligned}$$

Hence the resulting scaling transformation about the center (a, b) is also represented by a 3×3 matrix, namely $M_3 M_2 M_1$. We see that composition of transformations corresponds to matrix multiplication.

Another important affine transformation of the plane is a *rotation* about the point $(0, 0)$. To derive its equations, recall the polar coordinates of a point. Every point X in $\mathbb{R}^2$ is represented by two numbers r and φ , where $r \geq 0$ is the distance of the point from the origin O , and $\varphi \in \langle 0, 2\pi \rangle$ is the oriented angle from the positive x -axis to the ray $\overrightarrow{OX}$. Cartesian and polar coordinates are related by

$$\begin{aligned} x &= r \cos \varphi \\ y &= r \sin \varphi. \end{aligned}$$

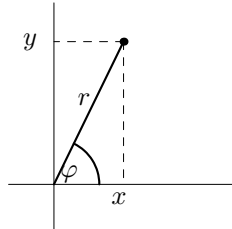


FIGURE 1. Cartesian and polar coordinates of a point

After a rotation through an angle α , the polar coordinates of the transformed point are $r, \varphi + \alpha$. Hence, for the Cartesian coordinates we obtain

$$\begin{aligned} x' &= r \cos(\varphi + \alpha) = r \cos \varphi \cos \alpha - r \sin \varphi \sin \alpha = x \cos \alpha - y \sin \alpha \\ y' &= r \sin(\varphi + \alpha) = r \cos \varphi \sin \alpha + r \sin \varphi \cos \alpha = x \sin \alpha + y \cos \alpha. \end{aligned}$$

Thus rotation about $(0, 0)$ through an angle α has the matrix form

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Among the special affine transformations, let us also mention reflection and shear. A *reflection* is an axial symmetry; reflection in the x -axis is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and, similarly, reflection in the y -axis by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every reflection is an orientation-reversing transformation of the plane.

By the *orientation of the plane* we mean both the orientation of objects in the plane and the orientation of the coordinate system. A coordinate system is *positively oriented* if the positive y -axis is obtained from the positive x -axis by a counterclockwise rotation through a right angle; otherwise it is *negatively oriented*. If the coordinate axes are not required to be perpendicular, the coordinate system is positively oriented when the positive y -axis is obtained from the positive x -axis by a counterclockwise rotation through an angle in $(0, \pi)$.

Orientation is also associated with polygons. For example, a triangle ABC is positively oriented if its vertices are listed counterclockwise. Other simple polygons are oriented in the same way.

Orientation also plays a role in measuring and specifying angles. For example, a rotation through a given angle is always taken in the positive sense of the coordinate system; for a positively oriented system this is counterclockwise.

Saying that a transformation reverses orientation means that the image of a positively oriented coordinate system is negatively oriented, and vice versa. After applying a transformation we usually introduce a new coordinate system having the original (typically positive) orientation. The new coordinates (x', y') are coordinates with respect to this new system. One must nevertheless remember that the orientation of objects in the plane (for example, triangles) has been reversed.

Finally, an x -*shear* is a transformation that preserves the y -coordinate while modifying the x -coordinate linearly according to the distance from the x -axis. Its matrix is

$$\begin{pmatrix} 1 & s_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, a y -*shear* is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ s_y & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, every affine transformation of the plane can be represented by a 3×3 matrix with six degrees of freedom:

$$(1) \quad \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

This matrix is nonsingular because its upper-left 2×2 submatrix is nonsingular. The inverse transformation is therefore represented by the inverse of the 3×3 matrix in (1).

Proposition 2.2. *An affine transformation of $\mathbb{R}^2$ is uniquely determined by the images of three non-collinear points.*

Thus, to determine the matrix of a general affine transformation (one that is not simply a combination of translations, rotations, and the like), we may choose three non-collinear points and specify their images. All required constants c_{ij} can then be computed.

Example 2.3. Find the equations of the affine transformation that maps the house in the figure to its shadow.

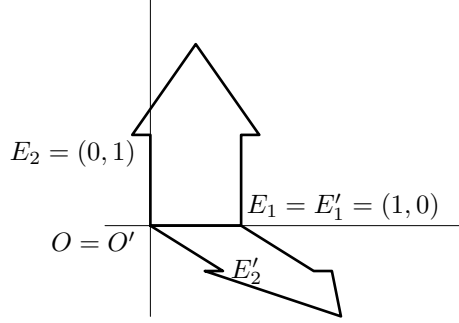


FIGURE 2. Affine transformation

Solution. The transformation fixes the points O and E_1 , while E_2 is mapped to E_2' with coordinates $(0.8, -0.5)$. The point O and its image give linear conditions on the constants c_{ij} :

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{10} \\ c_{21} & c_{22} & c_{20} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and hence $c_{10} = c_{20} = 0$. Similarly, substituting the coordinates of E_1 and its image E_1' into (1) gives $c_{11} = 1$ and $c_{21} = 0$. Finally, from the coordinates of E_2 and its image E_2' we obtain $c_{12} = 0.8$ and $c_{22} = -0.5$. The matrix of the required mapping is

$$\begin{pmatrix} 1 & 0.8 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

An important submatrix of the matrix in (1) is its upper-left 2×2 submatrix

$$A = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

From the definition, $\det A \neq 0$. The following properties hold:

Proposition 2.4. *The affine transformation (1) reverses the orientation of the plane if and only if $\det A < 0$.*

Proposition 2.5. *The affine transformation (1) is Euclidean (distance-preserving) if and only if $AA^T = I_2$.*

3. TRANSFORMATIONS IN $\mathbb{R}^3$

As in the plane, an affine transformation of $\mathbb{R}^3$ is described by the linear equations

$$(2) \quad \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{10} \\ c_{21} & c_{22} & c_{23} & c_{20} \\ c_{31} & c_{32} & c_{33} & c_{30} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}, \quad \text{where} \quad \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} \neq 0.$$

Proposition 3.1. *An affine transformation of $\mathbb{R}^3$ is uniquely determined by the images of four non-coplanar points.*

Again denoting the 3×3 submatrix $(c_{ij})_{i,j=1}^3$ by A , we obtain properties analogous to those for transformations of the plane:

Proposition 3.2. *The affine transformation (2) is Euclidean (distance-preserving) if and only if $AA^T = I_3$.*

Proposition 3.3. *The affine transformation (2) reverses the orientation of space if and only if $\det A < 0$.*

We use the right-hand rule to define positive orientation in space: the fingers (except the thumb) indicate the direction of rotation from the x -axis toward the y -axis, while the thumb points in the positive z -direction. This is also called a right-handed orientation. The opposite orientation is called left-handed or negative.

Translation and *scaling* in space are described analogously to the planar case. There are three basic *reflections* in space, one in each coordinate plane, and their matrices are straightforward to write down.

There are six basic affine *shear* transformations in space. For each one, we specify which coordinate plane remains fixed and the axis direction along which the shear is applied. For example, a shear in the y -direction with the yz -plane fixed is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A *rotation* in space means rotation about a chosen axis, regarded as an oriented line. In a right-handed coordinate system, the direction of rotation is determined by the right-hand rule: point the thumb along the orientation of the rotation axis, and the curled fingers indicate the positive direction of rotation. In particular, about the z -axis the positive rotation is from the x -axis toward the y -axis; about the x -axis it is from y toward z ; and about the y -axis it is from z toward x .

Consider rotation about the z -axis through an angle α . Restricted to the xy -plane, or to any plane parallel to it, this is simply a planar rotation about the origin $(0,0)$. Hence

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 & 0 \\ \sin \alpha & \cos \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}.$$

Rotations about the other two coordinate axes are obtained by the cyclic permutation $x \rightsquigarrow y \rightsquigarrow z \rightsquigarrow x$. For example, the matrix of rotation about the x -axis is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha & 0 \\ 0 & \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

s

Let us now find the matrix of a rotation about an arbitrary axis that, for the moment, passes through the origin. As in the planar case, we reduce the problem to a familiar one by first applying known rotations to the scene so that the rotation axis coincides with a coordinate axis.

Since the rotation axis passes through $(0,0,0)$, it is uniquely determined by one additional point $X = (x,y,z)$. Let this point (more precisely, its position vector) also determine the orientation of the axis. First find the spherical coordinates r, φ, ψ of X .

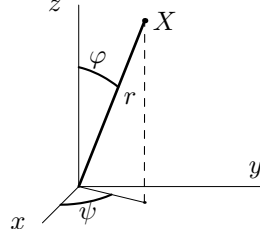


FIGURE 3. Spherical coordinates of a point

The relation between spherical and Cartesian coordinates is

$$\begin{aligned}x' &= r \sin \varphi \cos \psi \\y' &= r \sin \varphi \sin \psi \\z' &= r \cos \varphi.\end{aligned}$$

A rotation about the axis OX through an angle α can then be composed, for example, from the following transformations:

- (1) rotation through $-\psi$ about the z -axis (bringing the rotation axis into the xy -plane),
- (2) rotation through $-\varphi$ about the y -axis (making the rotation axis coincide with the z -axis),
- (3) rotation through α about the z -axis,
- (4) rotation through φ about the y -axis,
- (5) rotation through ψ about the z -axis,

Finally, let the rotation axis be an arbitrary oriented line. We again reduce the problem to a sequence of known transformations by translating the axis so that it passes through the origin. Let P be a point on the rotation axis. The required transformations are

- (1) translation by $O - P$,
- (2) rotation through α about an axis passing through $O = (0, 0, 0)$,
- (3) translation by $P - O$.

4. ROTATIONS IN SPACE USING QUATERNIONS

The set of quaternions is the set $\mathbb{H}$ of elements of the form

$$q = q_0 + q_1i + q_2j + q_3k, \quad c_i \in \mathbb{R},$$

where i, j, k are called the *imaginary units*. Every quaternion can be multiplied by a real scalar r :

$$r(q_0 + q_1i + q_2j + q_3k) = rq_0 + rq_1i + rq_2j + rq_3k.$$

We also define addition and multiplication on this set. Addition is performed componentwise:

$$(q_0 + q_1i + q_2j + q_3k) + (q'_0 + q'_1i + q'_2j + q'_3k) = (q_0 + q'_0) + (q_1 + q'_1)i + (q_2 + q'_2)j + (q_3 + q'_3)k.$$

Thus $\mathbb{H}$ is a four-dimensional vector space over $\mathbb{R}$. Quaternion multiplication is defined by the products of the imaginary units. Set

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

It can be verified (for example by checking all triples of imaginary units) that this multiplication is associative. Quaternion multiplication is, however, not commutative. The multiplicative identity is $1 = 1 + 0i + 0j + 0k$. Finally, every nonzero quaternion q has an inverse $q^{-1} \in \mathbb{H}$ satisfying $qq^{-1} = q^{-1}q = 1$. Apart from scalar multiplication by real numbers, the essential property distinguishing $\mathbb{H}$ from a field is precisely the noncommutativity of multiplication.

The *norm* of a quaternion $q = q_0 + q_1i + q_2j + q_3k$ is the real number

$$|q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

For a quaternion $q \in \mathbb{H}$, $q = s + v_1i + v_2j + v_3k$, $s, v_i \in \mathbb{R}$, we call s its *scalar part* and $\mathbf{v} = v_1i + v_2j + v_3k$ its *vector part*. We also write $q = s + \mathbf{v}$, or briefly $q = (s, \mathbf{v})$. If the scalar part s is zero, q is called a *pure imaginary quaternion*.

We encode each point of $\mathbb{R}^3$ as a quaternion: a point P with coordinates (p_1, p_2, p_3) is represented by $q_P = p_1i + p_2j + p_3k$. Thus we obtain a mapping $\mathbb{R}^3 \rightarrow \mathbb{H}$. This is a bijection between $\mathbb{R}^3$ and the set of all pure imaginary quaternions: every quaternion with zero scalar part represents exactly one point of $\mathbb{R}^3$.

We also associate rotations about arbitrary axes through $(0, 0, 0)$ with unit quaternions. Let (u_1, u_2, u_3) be a unit vector specifying the rotation axis, and let α be the rotation angle. The transformation is represented by the quaternion

$$q_{\mathbf{u}, \alpha} = \cos \frac{\alpha}{2} + (\sin \frac{\alpha}{2})(u_1i + u_2j + u_3k).$$

Conversely, every unit quaternion determines a rotation angle α and, when $\alpha \neq 0$, a unit vector $\mathbf{u}$ specifying the corresponding rotation axis.

Theorem 4.1. *Under rotation through angle α about the axis through the origin in the direction of the unit vector $\mathbf{u}$, the point $P \in \mathbb{R}^3$ (represented by q_P) is mapped to the point represented by*

$$q_{\mathbf{u}, \alpha} q_P q_{\mathbf{u}, \alpha}^{-1}.$$

We now rewrite the product in the theorem using only familiar vector operations: the dot product, the cross product, and scalar multiplication. Let q, q' be pure imaginary quaternions, $q = q_1i + q_2j + q_3k$ and $q' = q'_1i + q'_2j + q'_3k$. We may regard them as vectors in $\mathbb{R}^3$ with coordinates (q_1, q_2, q_3) and (q'_1, q'_2, q'_3) , respectively. We denote the dot product by $\langle \cdot, \cdot \rangle$ and the cross product by $\times$, so

$$\begin{aligned} \langle q, q' \rangle &= q_1q'_1 + q_2q'_2 + q_3q'_3, \\ q \times q' &= (q_2q'_3 - q_3q'_2, q_3q'_1 - q_1q'_3, q_1q'_2 - q_2q'_1). \end{aligned}$$

For $q, q' \in \mathbb{H}$, the notation qq' continues to denote associative quaternion multiplication.

The following properties follow directly by expanding the coordinates:

Lemma 4.2. *If $|q| = 1$ and $q = s + \mathbf{v}$ (s is the scalar part and $\mathbf{v}$ the vector part), then $q^{-1} = s - \mathbf{v}$.*

Lemma 4.3. *If q, q' are pure imaginary quaternions, then*

$$qq' = -\langle q, q' \rangle + q \times q'.$$

Thus

$$q_{\mathbf{u}, \alpha} q_P q_{\mathbf{u}, \alpha}^{-1} = (s + \mathbf{v})q_P(s - \mathbf{v}) = \dots = s^2q_P + 2s\mathbf{v} \times q_P + \langle q_P, \mathbf{v} \rangle \mathbf{v} + \mathbf{v} \times (\mathbf{v} \times q_P).$$

An advantage of using quaternions for rotations in $\mathbb{R}^3$ is that they avoid repeated expensive evaluations of trigonometric functions. Instead, they use vector operations, which are often implemented efficiently.

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