Computer graphics III – Path tracing

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Tracing paths from the camera

```
renderImage()
{
  for all pixels
  {
    Color pixelColor = (0,0,0);
    for k = 1 to N
    {
      ωk := random direction through the pixel
      pixelColor += getLi(camPos, ωk)
    }
    pixelColor /= N;
    writePixel(pixelColor);
  }
}
```

Path tracing, v. zero (recursive form)

getLi (x, ω):

```
y = traceRay(x, ω)
return
        Le(y, −ω) +                    // emitted radiance
        Lr (y, −ω)                     // reflected radiance
```

Lr(x, ω):

```
ω′ = genUniformHemisphereRandomDir( n(x) )
return 2π * brdf(x, ω, ω′)  * dot(n(x), ω′) * getLi(x, ω′)
```

Path Tracing – Loop version

```
getLi(x, w)
{
    Color thrput = (1,1,1)
    Color accum  = (0,0,0)
    while(1)
    {
        hit = NearestIntersect(x, w)
        if no intersection
                return accum + thrput * bgRadiance(x, w)
        if isOnLightSource(hit)
                accum += thrput * Le(hit.pos, -w)
        ρ = reflectance(hit.pos, -w)
        if rand() < ρ // russian roulette - survive (reflect)
                wi := SampleDir(hit)
                thrput *= fr(hit.pos, wi, -w) * dot(hit.n, wi) / (ρ * pdf(wi))
                x  := hit.pos
                w  := wi
        else // absorb
            break;
    }
    return accum;
}
```

Terminating paths – Russian roulette

```
getLi(x, w)
{
    Color thrput = (1,1,1)
    Color accum  = (0,0,0)
    while(1)
    {
        hit = NearestIntersect(x, w)
        if no intersection
            return accum + thrput * bgRadiance(x, w)
        if isOnLightSource(hit)
            accum += thrput * Le(hit.pos, -w)
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            wi := SampleDir(hit)
            thrput *= fr(hit.pos, wi, -w) * dot(hit.n, wi) / (ρ * pdf(wi))
            x  := hit.pos
            w  := wi
        else // absorb
            break;
    }
    return accum;
}
```

Terminating paths – Russian roulette

- Continue the path with probability q
- Multiply weight (throughput) of surviving paths by $1 / q$

$$Z = \begin{cases} Y / q & \text{if } \xi < q \\ 0 & \text{otherwise} \end{cases}$$

- RR is unbiased!

$$E[Z] = \frac{E[Y]}{q} \cdot q + 0 \cdot (1 - q) = E[Y]$$

Survival probability

- It makes sense to use the surface reflectivity ρ as the survival probability
  - If the surface reflects only 30% of energy, we continue with the probability of 30%. That's in line with what happens in reality.
- What if we cannot calculate ρ? Then there's a convenient alternative:
  1. First sample a random direction ω_i according to $p(\omega_i)$
  2. Use the sampled ω_i it to calculate the survival probability as

$$q_{\text{survival}} = \min\left\{1,\ \frac{f_r(\omega_i \rightarrow \omega_o)\cos\theta_i}{p(\omega_i)}\right\}$$

Direction sampling

```
getLi(x, w)
{
    Color thrput = (1,1,1)
    Color accum  = (0,0,0)
    while(1)
    {
        hit = NearestIntersect(x, w)
        if no intersection
                return accum + thrput * bgRadiance(x, w)
        if isOnLightSource(hit)
                accum += thrput * Le(hit.pos, -w)
        ρ = reflectance(hit.pos, -w)
        if rand() < ρ // russian roulette - survive (reflect)
                wi := SampleDir(hit)
                thrput *= fr(hit.pos, wi, -w) * dot(hit.n, wi) / (ρ * pdf(wi))
                x  := hit.pos
                w  := wi
        else // absorb
            break;
    }
    return accum;
}
```

Direction sampling

- We usually sample the direction ω_i from a pdf similar to

$$f_r(\omega_i, \omega_o) \cos \theta_i$$

  - Ideally, we would want to sample proportionally to the integrand itself

$$L_i(\omega_i) f_r(\omega_i, \omega_o) \cos \theta_i,$$

    but this is difficult, because we do not know L_i upfront. With some precomputation, it is possible to use a rough estimate of L_i for sampling [Jensen 95, Vorba et al. 2014], cf. “guiding”.

No incoming radiance information [Vorba et al. 2014]

“Guiding” by incoming radiance [Vorba et al. 2014]

BRDF importance sampling

- Let's see what happens when the pdf is **exactly proportional** to $f_r(\omega_i, \omega_o) \cos \theta_i$?

$$p(\omega_i) \propto f_r(\omega_i \rightarrow \omega_o) \cdot \cos\theta_i$$

- Normalization (recall that a pdf must integrate to 1)

$$p(\omega_i) = \frac{f_r(\omega_i \rightarrow \omega_o) \cdot \cos\theta_i}{\int_{H(\mathbf{x})} f_r(\omega_i \rightarrow \omega_o) \cdot \cos\theta_i \, \mathrm{d}\omega_i}$$

The normalization factor is nothing but the reflectance ρ

BRDF IS in a path tracer

- Throughput update for a general pdf

```
...
thrput *= fr(.) * dot(.) / ( ρ * p(wi) )
```

- A pdf that is exactly proportional to BRDF * cos keeps the throughput constant because the different terms cancel out!

$$p(\omega_i) = f_r(\omega_i \rightarrow \omega_o) \cdot \cos\theta_i \;/\; \rho$$

```
...
thrput *= 1
```

Direct illumination calculation in a path tracer

Direct illumination: Two strategies

- At each path vertex **x**, we are calculating **direct illumination**
  - i.e. radiance reflected from a point **x** on a surface exclusively due to the light coming directly from the sources

- Two sampling strategies
  1. **BRDF-proportional sampling**
  2. **Light source area sampling**

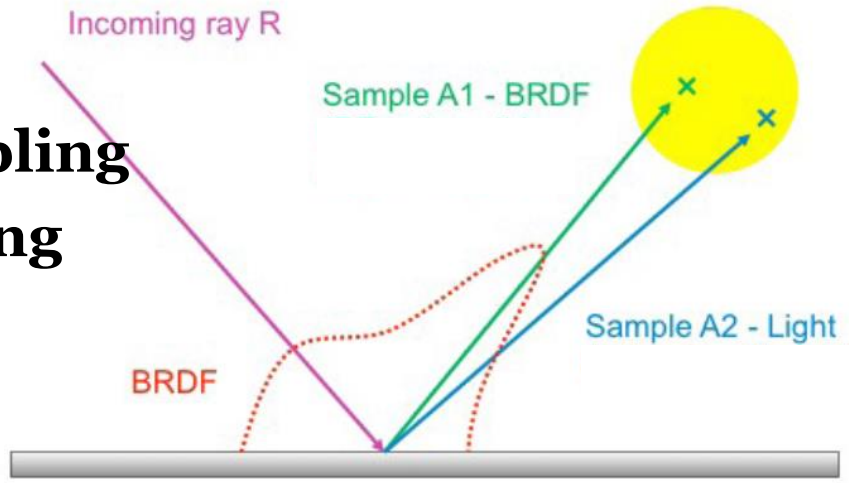


Image: Alexander Wilkie

Direct illumination

- The basic version of Path Tracer is not applicable in practice at all. The probability of the beam hitting the point light source is zero and low for small light sources.
- Convergence slows and variance increases. The figure shows the result we achieve in a scene with a single small light source.
- The solution is to determine the reflected radiance at a given point caused by direct lighting.

Image: Alexander Wilkie

Direct illumination: BRDF sampling

- We use the angular form of the reflection equation and are only interested direct illumination.
- To estimate the emitted radiance L_r, we use only the emitted radiance at the point $\mathbf{x}$ of impact of the beam in the direction ω_i:

$$L_r(\mathbf{x}, \omega_o) = \int_{H(\mathbf{x})} L_e(r(\mathbf{x}, \omega_i), -\omega_i) \cdot f_r(\mathbf{x}, \omega_i, \omega_o) \cdot \cos\theta_i \, \mathrm{d}\omega_i$$

- To estimate this integral, we use MC estimator:

$$\hat{L}_r(\mathbf{x}, \omega_o) = \frac{1}{N} \sum_{k=1}^{N} \frac{L_e(r(\mathbf{x}, \omega_{i,k}), -\omega_{i,k}) \cdot f_r(\mathbf{x}, \omega_{i,k}, \omega_o) \cdot \cos\theta_{i,k}}{p(\omega_{i,k})}$$

Direct illumination: light sampling

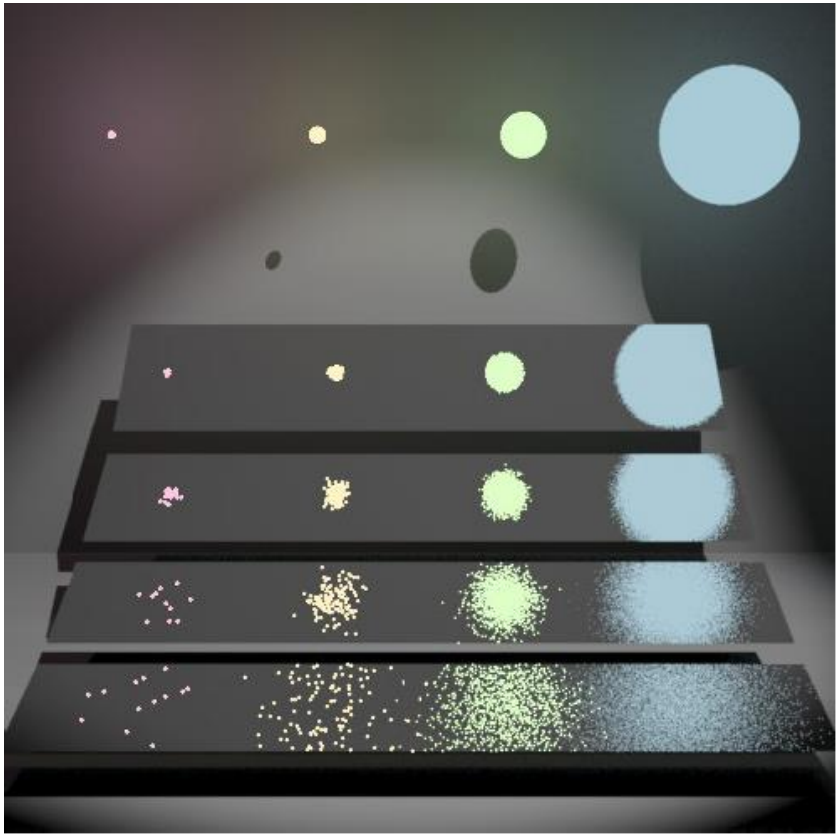
- We use the area form of the reflection equation and are only interested direct illumination.
- To estimate the emitted radiance L_r, we use only the emitted radiance at the point $\mathbf{x}$ and integrate across the surface A of the light source:

$$L_r(\mathbf{x}, \omega_o) = \int_A L_e(\mathbf{y} \to \mathbf{x}) \cdot f_r(\mathbf{y} \to \mathbf{x} \to \omega_o) \cdot V(\mathbf{y} \leftrightarrow \mathbf{x}) \cdot G(\mathbf{y} \leftrightarrow \mathbf{x}) \, \mathrm{d}A_{\mathbf{y}}$$

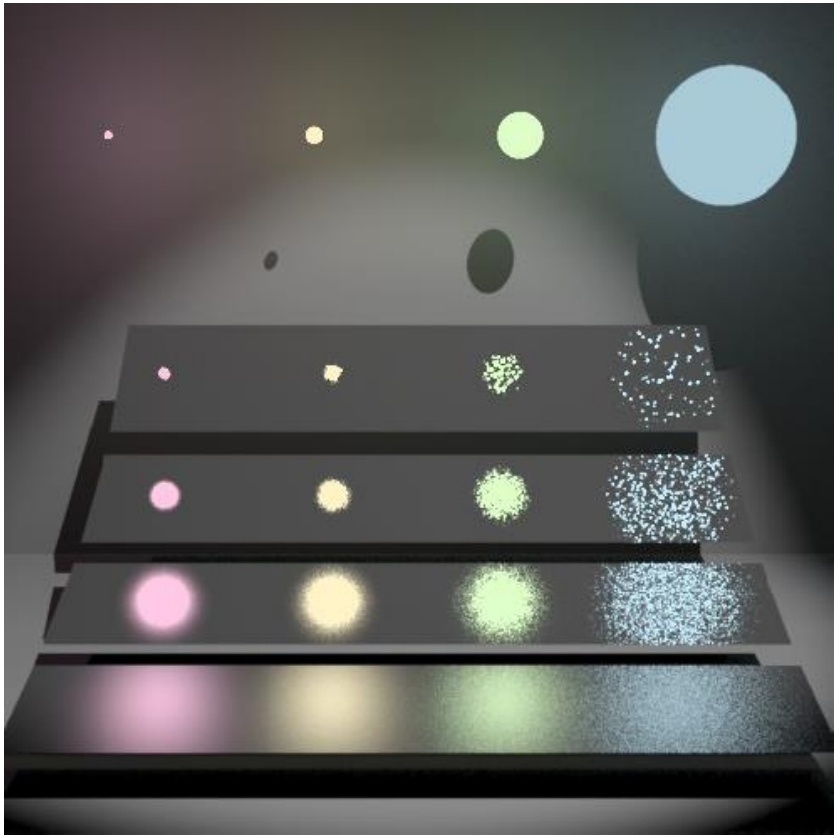
- We determine the value of this integral using the MC estimator. We choose a uniform probability density:

$$\hat{L}_r(\mathbf{x}, \omega_o) = \frac{|A|}{N} \sum_{k=1}^{N} L_e(\mathbf{y}_k \to \mathbf{x}) \cdot f_r(\mathbf{y}_k \to \mathbf{x} \to \omega_o) \cdot V(\mathbf{y}_k \leftrightarrow \mathbf{x}) \cdot G(\mathbf{y}_k \leftrightarrow \mathbf{x})$$

Direct illumination: Two strategies



BRDF proportional sampling



Light source area sampling

Images: Eric Veach

Direct illumination calculation using MIS

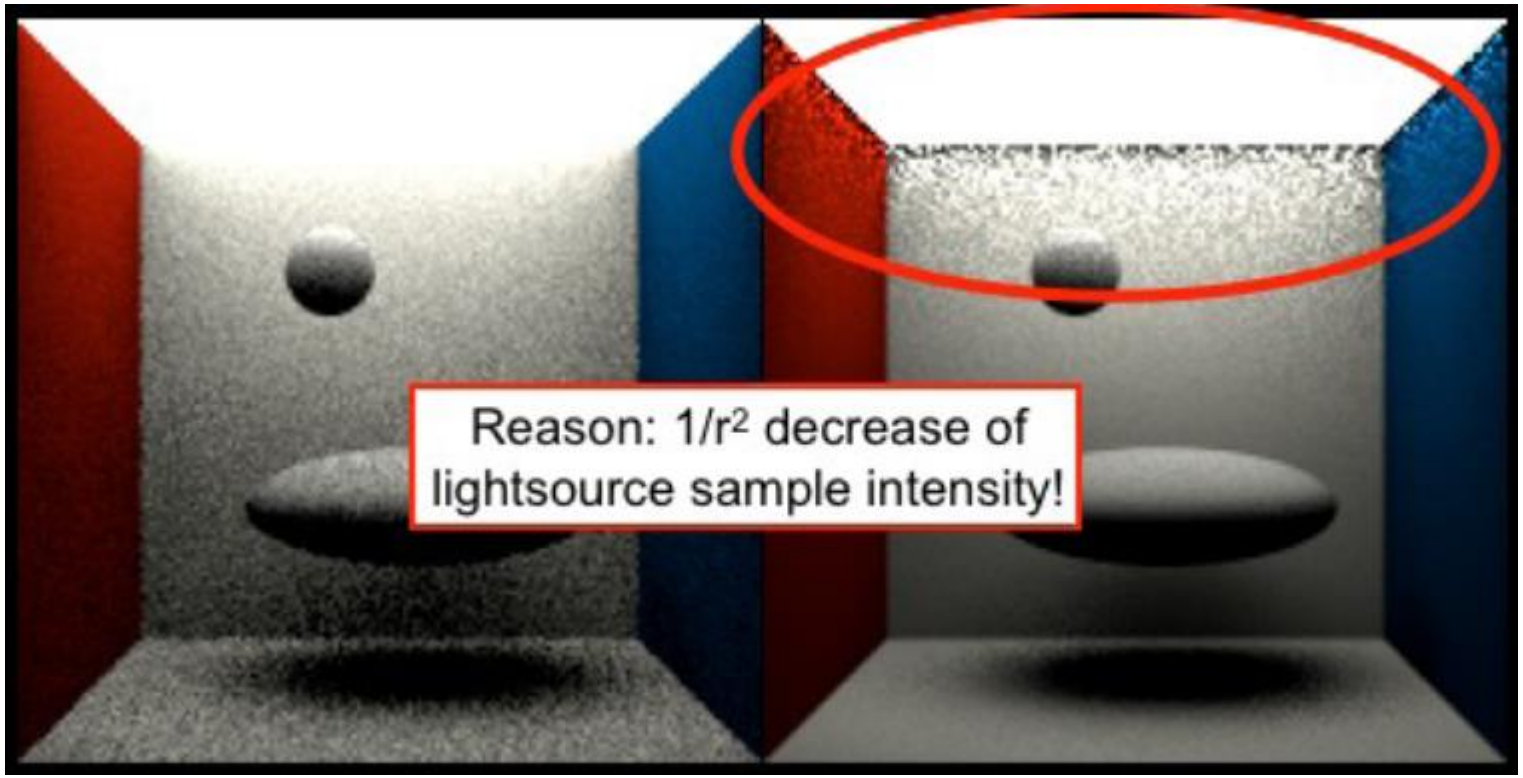
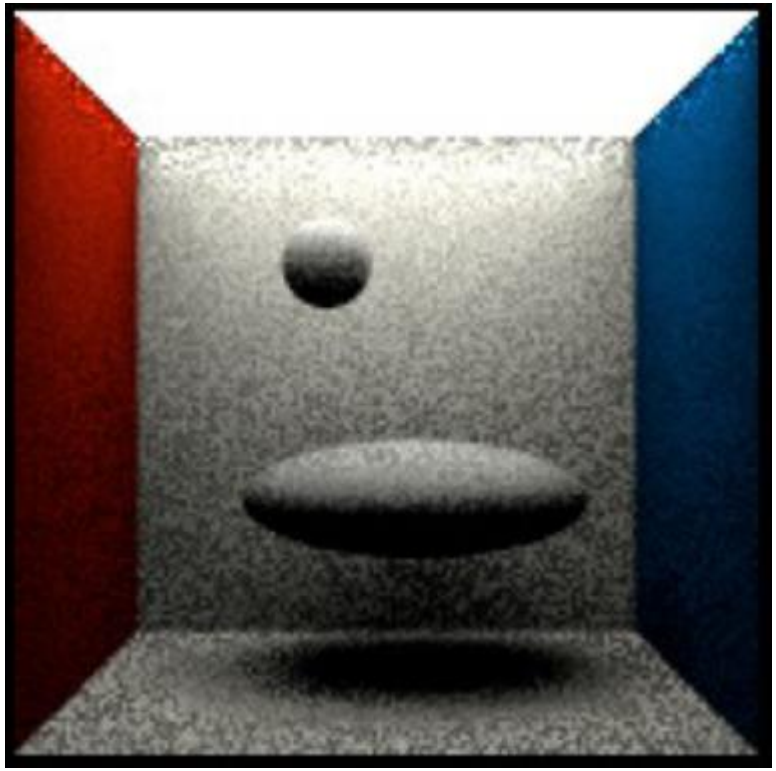


Image: Alexander Wilkie

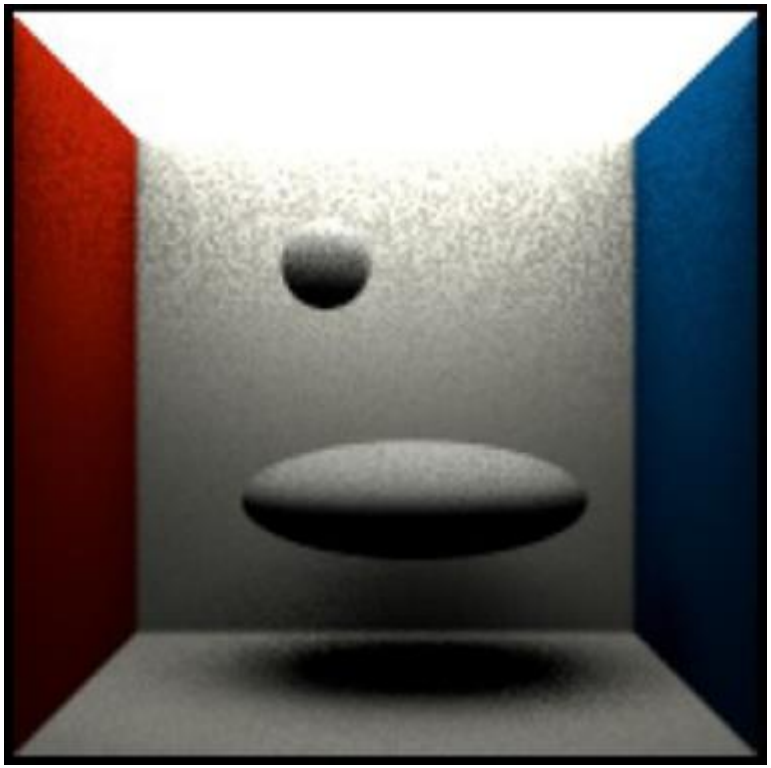
Sampling technique (pdf) p_1: BRDF sampling

Sampling technique (pdf) p_2: Light source area sampling

Combination



Arithmetic average
Preserves **bad** properties
of both techniques



Balance heuristic
Bingo!!!

Image: Alexander Wilkie

MIS weight calculation

Sample weight for BRDF sampling

$$w_1(\omega_j) = \frac{p_1(\omega_j)}{p_1(\omega_j) + p_2(\omega_j)}$$

PDF for BRDF sampling

PDF with which the direction ω_j would have been generated, if we used light source area sampling

PDFs

- **BRDF sampling: $p_1(\omega)$**
  - Depends on the BRDF, e.g. for a Lambertian BRDF:

$$p_1(\omega) = \frac{\cos\theta_{\mathbf{x}}}{\pi}$$

- **Light source area sampling: $p_2(\omega)$**

$$p_2(\omega) = \frac{1}{|A|} \frac{\|\mathbf{x}-\mathbf{y}\|^2}{\cos\theta_{\mathbf{y}}}$$

Conversion of the uniform pdf 1/|A| from the area measure (dA) to the solid angle measure (dω)

Where is the conversion factor coming from?

- Pdfs (unlike ordinary function) change under a change of coordinates. In general, it must always hold:

$$p(\omega)d\omega = p(\mathbf{y})dA$$

- And so

$$p(\omega) = p(\mathbf{y})\frac{dA}{d\omega}$$

conversion factor

The use of MIS in a path tracer

- For each path vertex:
  - Generate an explicit shadow ray for the technique p_2 (light source area sampling)
  - Secondary ray ω_{new} for technique p_1 (BRDF sampling)
    - One ray can be shared for the calculation of both **direct** and **indirect** illumination
    - But the MIS weight is – of course – applied only on the direct term (indirect illumination is added unweighted because there is no second technique to calculate it)

$$\hat{L}_r(\mathbf{x},\omega_o) = \frac{L_e(r(\mathbf{x},\omega_{new}),-\omega_{new})\cdot f_r(\mathbf{x},\omega_{new},\omega_o)\cdot\cos\theta_{new}}{\rho\cdot(p_1(\omega_{new})+p_2(\omega_{new}))} + \frac{L_e(\mathbf{y}\rightarrow\mathbf{x})\cdot f_r(\mathbf{y}\rightarrow\mathbf{x}\rightarrow\omega_o)\cdot G(\mathbf{y}\leftrightarrow\mathbf{x})}{p_1(\mathbf{y})+p_2(\mathbf{y})} + \frac{\hat{L}_r(r(\mathbf{x},\omega_{new}),-\omega_{new})\cdot f_r(\mathbf{x},\omega_{new},\omega_o)\cdot\cos\theta_{new}}{\rho\cdot p_1(\omega_{new})}$$

Dealing with multiple light sources

- Option 1:
  - Loop over all sources and send a shadow ray to each one
- Option 2:
  - Choose one source at random (with probability proportional to power)
  - Sample illumination only on the chosen light, divide the result by the probability of picking that light
  - (Scales better with many sources but has higher variance per path)
- Beware: The probability of choosing a light influences the sampling pdf and therefore also the MIS weights.