

Ray \times scene intersections

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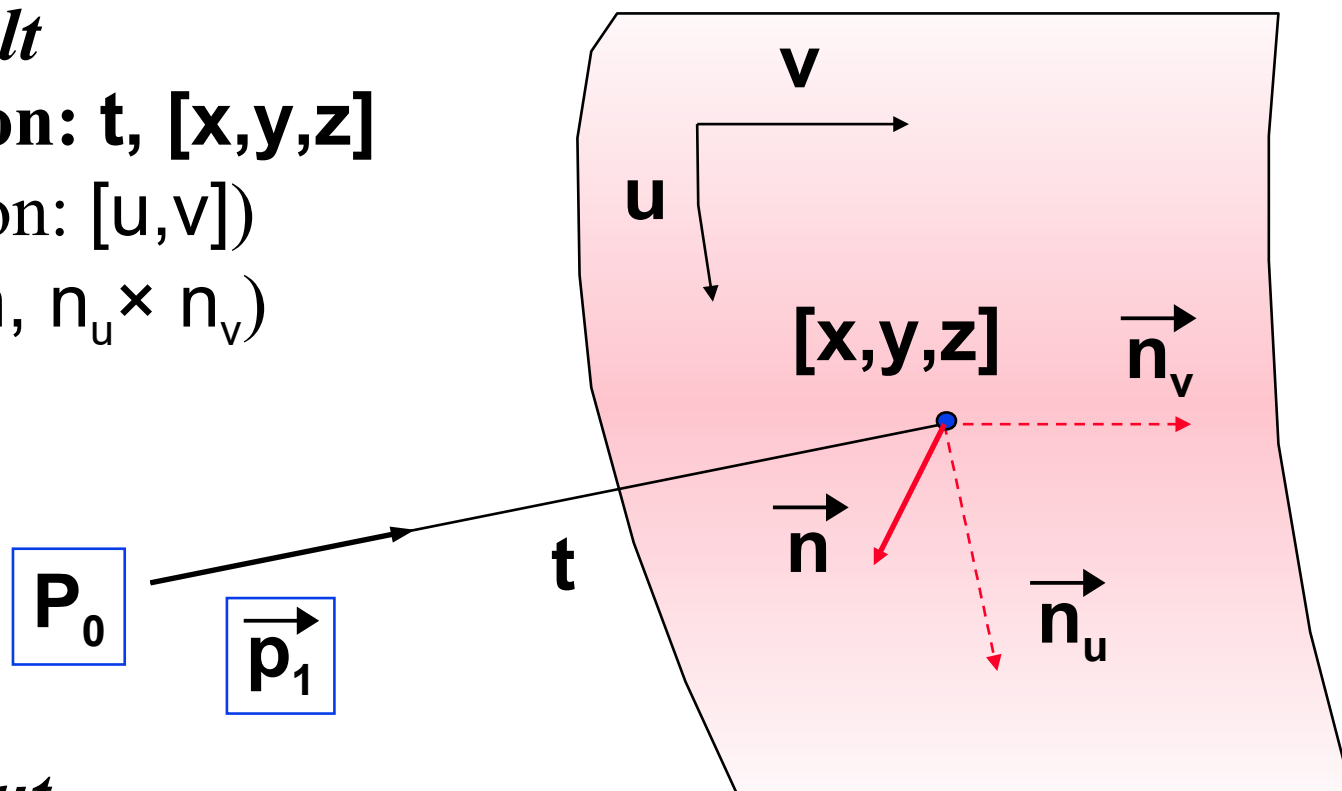
Ray × scene intersection

result

3D position: \mathbf{t} , $[\mathbf{x}, \mathbf{y}, \mathbf{z}]$

(2D position: $[\mathbf{u}, \mathbf{v}]$)

(Normal: \mathbf{n} , $\mathbf{n}_u \times \mathbf{n}_v$)



input

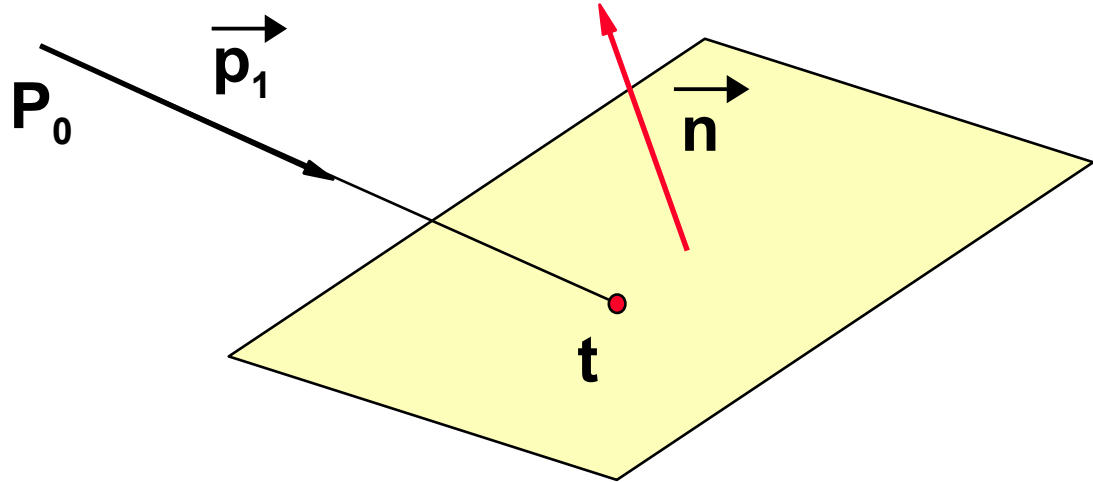
Ray: P_0, \vec{p}_1



Plane

ray:

$$P(t) = P_0 + t \cdot \vec{p}_1$$



plane:

$$\vec{n} = [x_n, y_n, z_n]$$

$$x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0$$

- intersection $t = -(\vec{n} \cdot P_0 + D) / (\vec{n} \cdot \vec{p}_1)$
- negative: $2\pm, 3^*$, positive: $5\pm, 6^*, 1/$
- computation of $[x, y, z]$: $3\pm, 3^*$

Inverse transformation on the plane

plane:

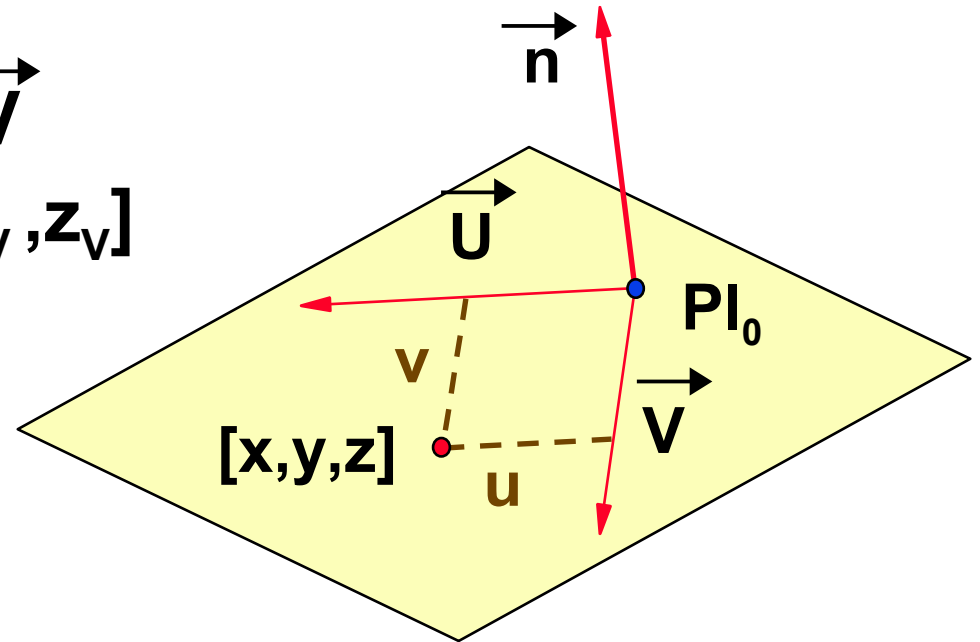
$$PI(u,v) = PI_0 + u \cdot \vec{U} + v \cdot \vec{V}$$

$$\vec{U} = [x_u, y_u, z_u], \vec{V} = [x_v, y_v, z_v]$$

$$\vec{n} = \vec{U} \times \vec{V}$$

input: $PI, \vec{U}, \vec{V}, [x,y,z]$

result: $[u,v]$



- linear system $\underline{u} \cdot x_u + \underline{v} \cdot x_v = x - PI_{0x}$
 $\underline{u} \cdot y_u + \underline{v} \cdot y_v = y - PI_{0y}$
- solution $[u,v]: 5\pm, 5^*, 2/$



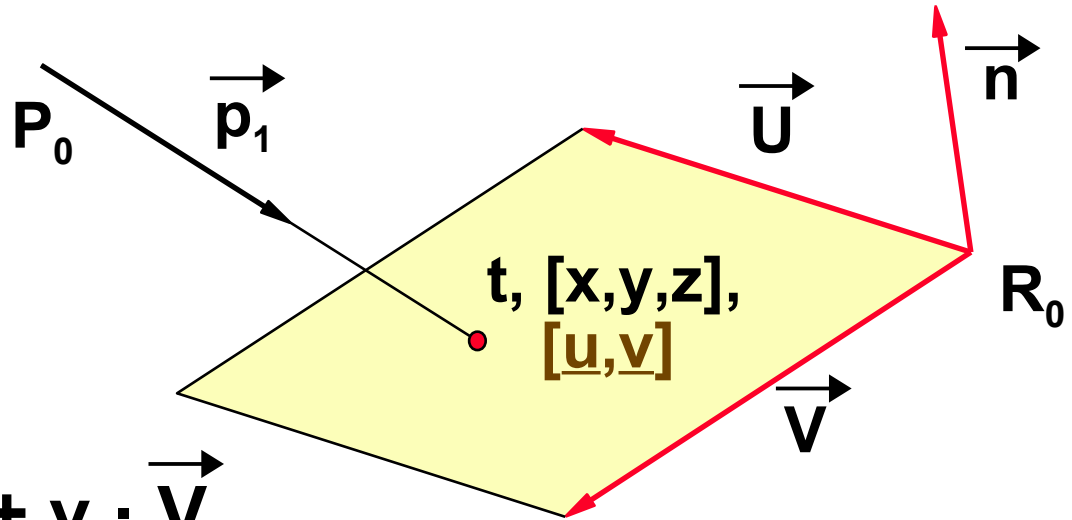
Parallelogram

ray:

$$P(t) = P_0 + t \cdot \vec{p}_1$$

parallelogram:

$$R(u,v) = R_0 + u \cdot \vec{U} + v \cdot \vec{V}$$
$$0 \leq u,v \leq 1$$



- computing $t, [x,y,z], [u,v]$, tests of u,v
- positive case total: $13\pm, 14^*, 3/, 4\leq$



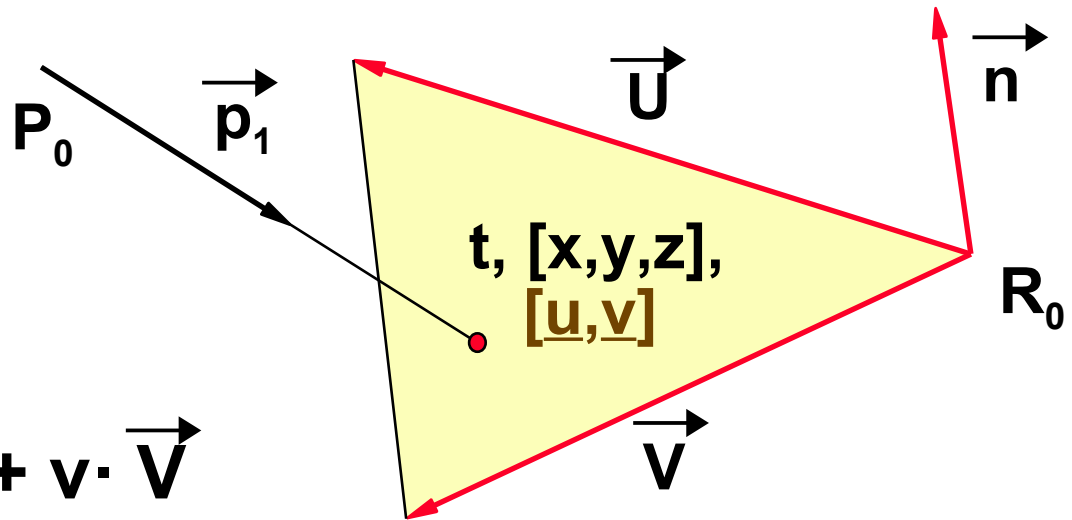
Triangle

ray:

$$P(t) = P_0 + t \cdot \vec{p}_1$$

triangle:

$$R(u,v) = R_0 + u \cdot \vec{U} + v \cdot \vec{V}$$
$$0 \leq u, v, u+v \leq 1$$



- computing $t, [x,y,z], [u,v]$, tests of u,v
- positive case total: $14\pm, 14^*, 3/, 3\leq$



General planar polygon

ray:

$$P(t) = P_0 + t \cdot \vec{p}_1$$

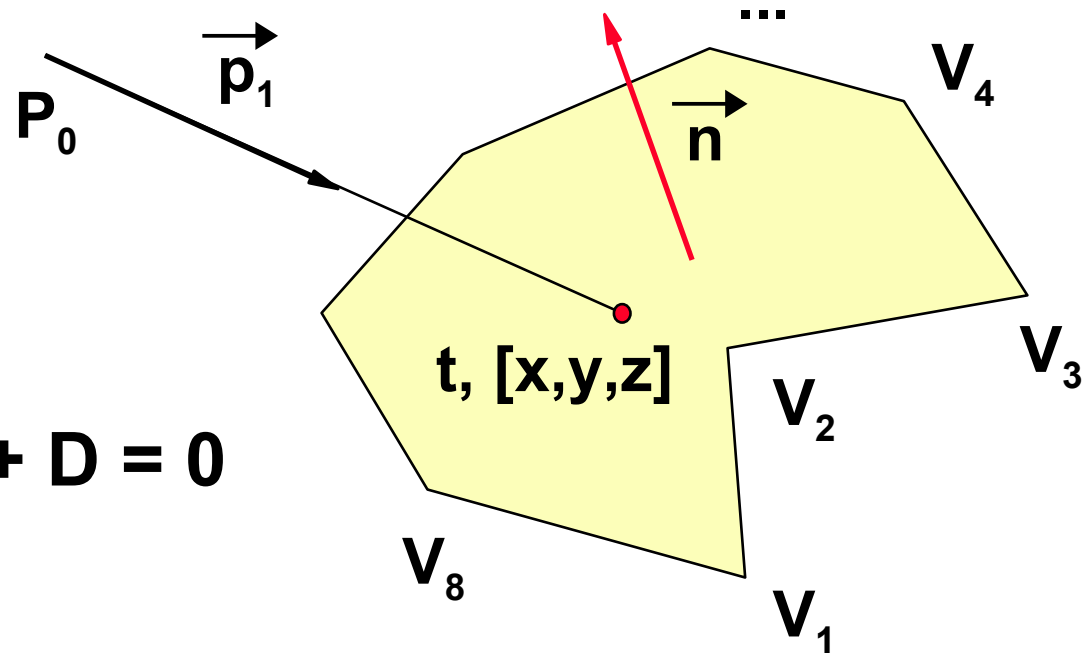
polygon plane:

$$\vec{n} = [x_n, y_n, z_n]$$

$$x \cdot x_n + y \cdot y_n + z \cdot z_n + D = 0$$

polygon vertices:

$$V_1, V_2, \dots, V_M$$



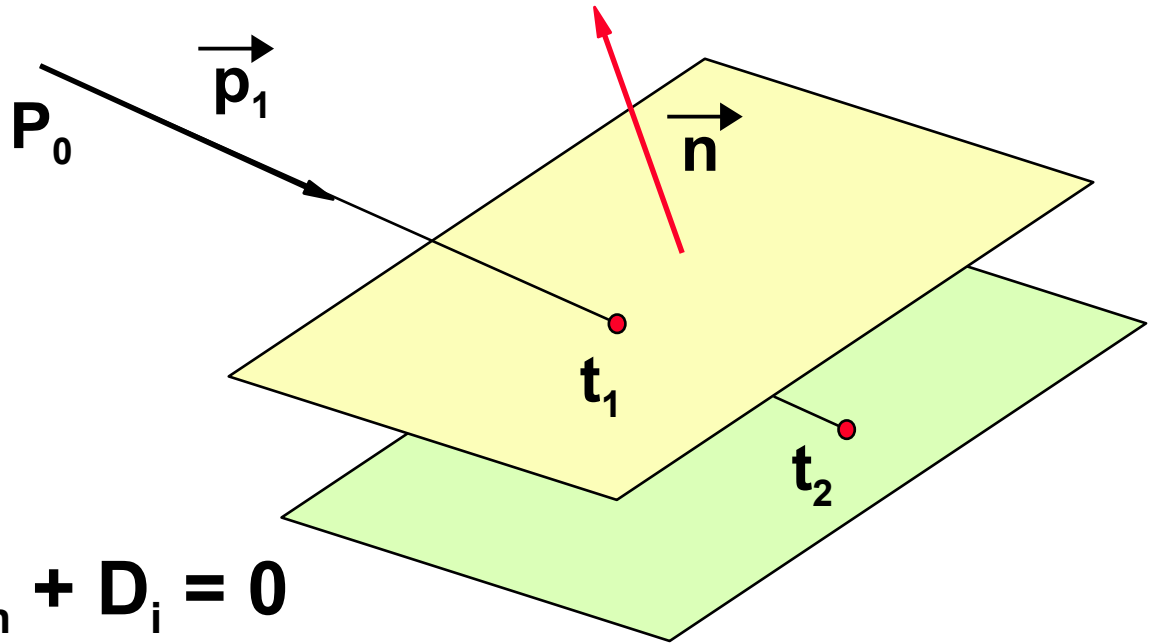
- computing $t, [x,y,z]$, planar test: **point** \times **polygon**
- intersection with the plane: **8** \pm , **9**^{*}, **1**/



Parallel planes

ray:

$$P(t) = P_0 + t \cdot \vec{p}_1$$



parallel planes:

$$\vec{n} = [x_n, y_n, z_n]$$

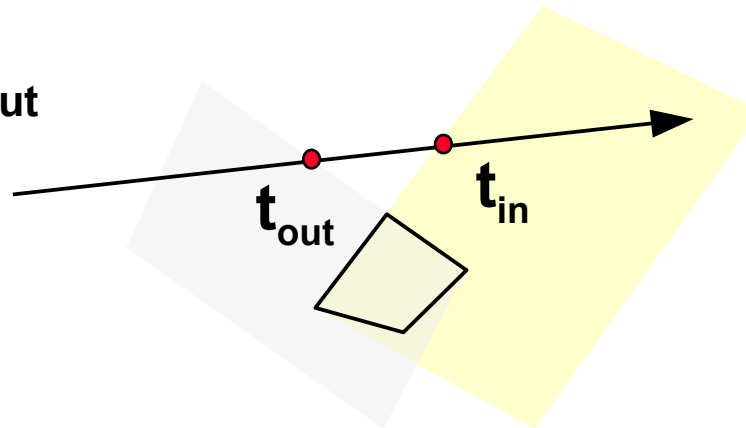
$$x \cdot x_n + y \cdot y_n + z \cdot z_n + D_i = 0$$

- intersections $t_i = -(\vec{n} \cdot P_0 + D_i) / (\vec{n} \cdot \vec{p}_1)$
- the 1st plane: $5\pm, 6^*, 1/$, every next one: $1\pm, 1/$



Convex polyhedron

- ♦ defined as an **intersection of K halfspaces**
 - at most **K** intersections ray vs. plane
 - **parallelism** of planes can be used – e.g. cuboid
- variables t_{in} , t_{out} initialized to $0, \infty$
- ray vs. one halfspace: $\langle t, \infty \rangle$ resp. $(-\infty, t]$
 $t_{in} = \max\{t_{in}, t\}$ resp. $t_{out} = \min\{t_{out}, t\}$
- early exit if $t_{in} > t_{out}$





Implicit surface

ray:

$$\mathbf{P}(t) = \mathbf{P}_0 + t \cdot \vec{\mathbf{p}}_1$$

implicit surface:

$$F(x,y,z) = 0$$

example:

$$(c - \cos ax) \cos z + (y + a \sin ax) \sin z + \cos a(x+z) = 0$$

- substitution $\mathbf{P}(t)$ into F : $F^*(t) = 0$
- finding roots of the function $F^*(t)$
 - sometimes only the **smallest positive root** is needed (the 1st intersection), for **CSG** we need **all roots**



Algebraic surface

ray:

$$\mathbf{P}(t) = \mathbf{P}_0 + t \cdot \vec{\mathbf{p}}_1$$

algebraic surface of degree d:

$$\mathbf{A}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \sum_{i,j,k=0}^{i+j+k \leq d} a_{ijk} \cdot \mathbf{x}^i \mathbf{y}^j \mathbf{z}^k = 0$$

example (toroid with radii a, b):

$$\mathbf{T}_{ab}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \left(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 - \mathbf{a}^2 - \mathbf{b}^2 \right)^2 - 4\mathbf{a}^2 \left(\mathbf{b}^2 - \mathbf{z}^2 \right)$$

- after substitution $\mathbf{P}(t)$ into \mathbf{A} : $\mathbf{A}^*(t) = 0$
- \mathbf{A}^* is a polynomial of degree d (at most)



Quadric (d=2)

general quadric:

$$\underline{\mathbf{x}^T \mathbf{Q} \mathbf{x} = 0}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{bmatrix}$$

after substitution of $\mathbf{P}(t)$:

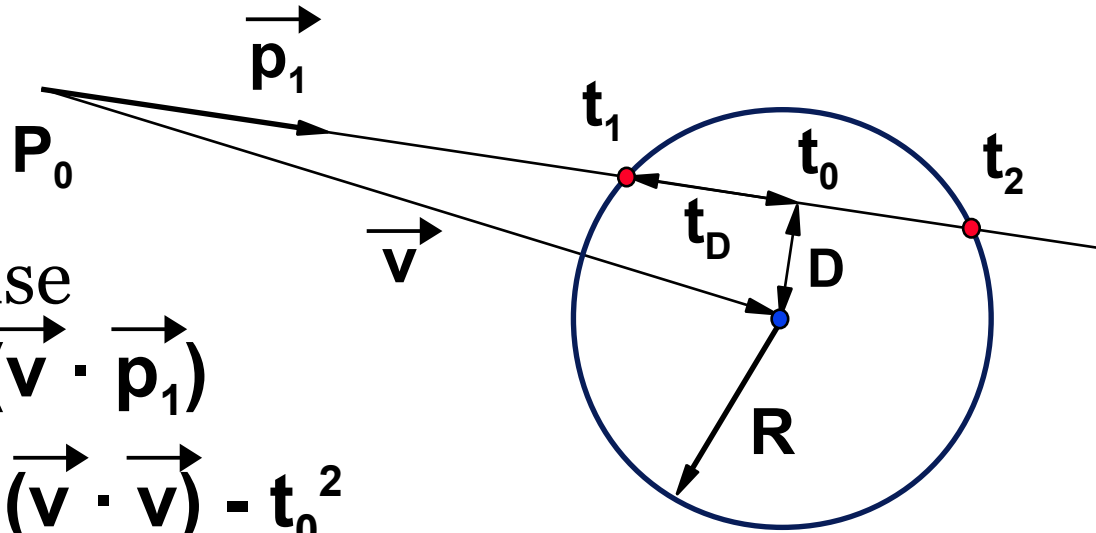
$$\underline{a_2 t^2 + a_1 t + a_0 = 0,}$$

$$\text{where } a_2 = \mathbf{P}_1^T \mathbf{Q} \mathbf{P}_1, \quad a_1 = 2\mathbf{P}_1^T \mathbf{Q} \mathbf{P}_0, \quad a_0 = \mathbf{P}_0^T \mathbf{Q} \mathbf{P}_0$$



Sphere (geometric solution)

$$\mathbf{P}(t) = \mathbf{P}_0 + t \cdot \vec{\mathbf{p}}_1$$



- center of the subtense

$$t_0 = (\vec{\mathbf{v}} \cdot \vec{\mathbf{p}}_1)$$

- distance $D^2 = (\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) - t_0^2$

- inclination $t_D^2 = R^2 - D^2$

- for $t_D^2 = 0$ there is one tangent point $\mathbf{P}(t_0)$

- for $t_D^2 > 0$ two intersections exist: $\mathbf{P}(t_0 \pm t_D)$

- negative case: $9 \pm, 6^*, 1 <$, positive addit.: $2 \pm, 1 \text{ sqrt}$



Cylinder and cone

unit cylinder and unit cone in basic position:

$$\underline{x^2 + y^2 - 1 = 0}$$

$$\underline{x^2 + y^2 - z^2 = 0}$$

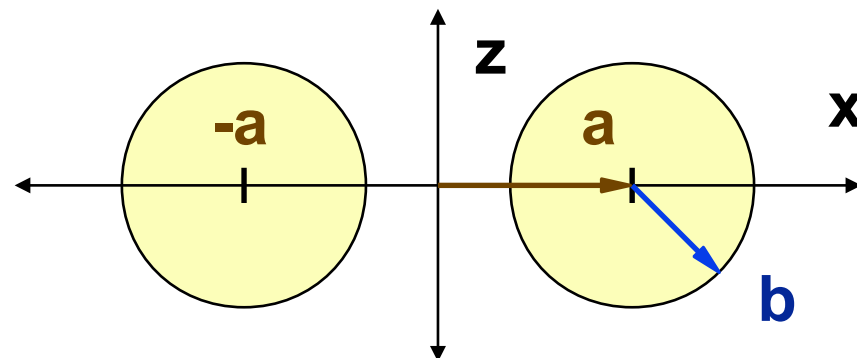
after substitution $P(t)$ for the cylinder:

$$\underline{t^2(x_1^2 + y_1^2) + 2t(x_0x_1 + y_0y_1) + x_0^2 + y_0^2 - 1 = 0}$$

after substitution $P(t)$ for the cone:

$$\underline{t^2(x_1^2 + y_1^2 - z_1^2) + 2t(x_0x_1 + y_0y_1 - z_0z_1) +} \\ \underline{+ x_0^2 + y_0^2 - z_0^2 = 0}$$

Toroid



Two circles in the xz plane:

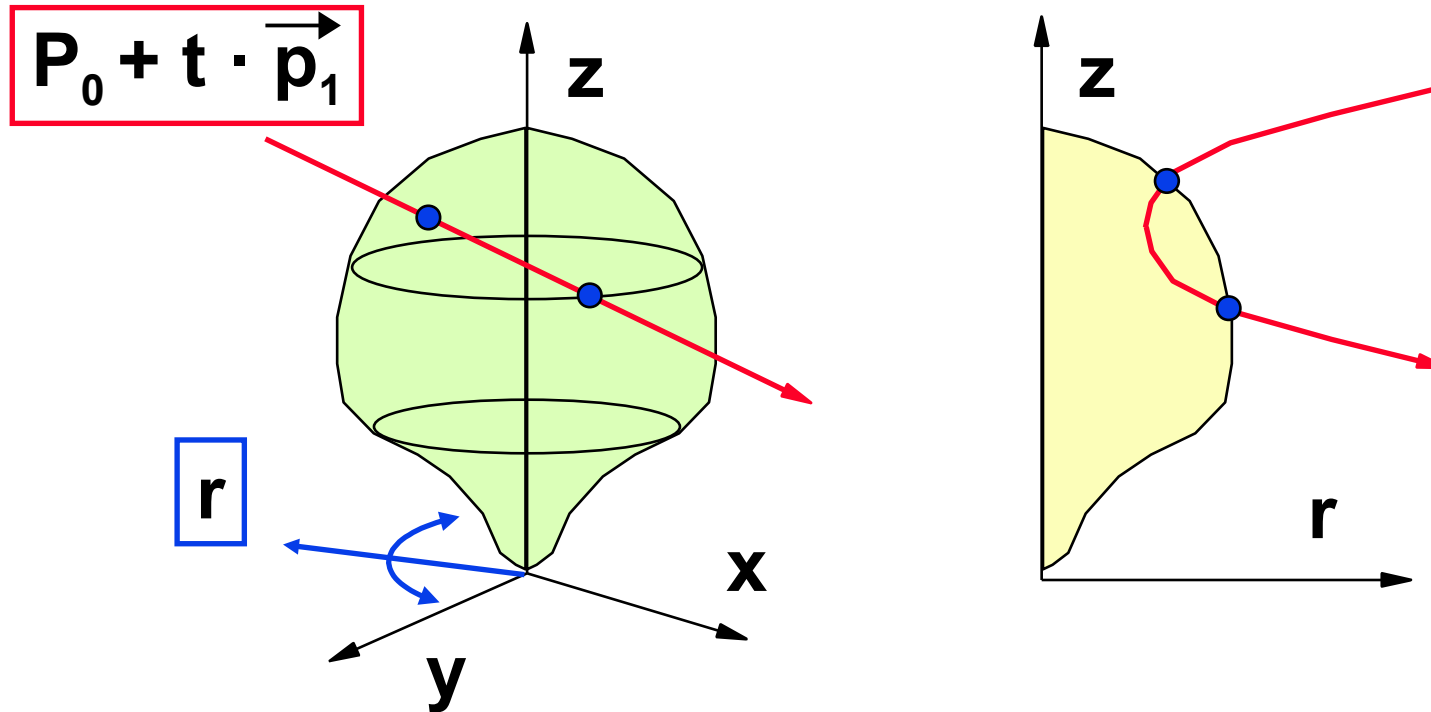
$$\left[(x - a)^2 + z^2 - b^2 \right] \cdot \left[(x + a)^2 + z^2 - b^2 \right] = 0$$
$$\left[x^2 + z^2 - (a^2 + b^2) \right]^2 = 4a^2(b^2 - z^2)$$

After substitution $b^2 = x^2 + y^2$ for x^2 – the 4th degree equation:

$$\left(x^2 + y^2 + z^2 - a^2 - b^2 \right)^2 - 4a^2(b^2 - z^2) = 0$$



Surface of revolution



equation of the ray in the rz plane:

$$r^2 = x^2 + y^2 = (x_0 + x_1 t)^2 + (y_0 + y_1 t)^2$$

$$z = z_0 + z_1 t$$



Ray in the rz plane

After elimination of t: $a r^2 + b z^2 + c z + d = 0$ (1)

$$a = -z_1^2$$

$$e = x_0 x_1 + y_0 y_1$$

$$b = x_1^2 + y_1^2$$

$$f = x_0^2 + y_0^2$$

$$c = 2(z_1 e - z_0 b)$$

$$d = z_0(z_0 b - 2 z_1 e) + f z_1^2$$

- after substitution of parametric curve $\mathbf{K}(\mathbf{s})$ into (1) we get an equation $\mathbf{K}^*(\mathbf{s}) = 0$
- \mathbf{K}^* has got a double degree (compared to \mathbf{K})

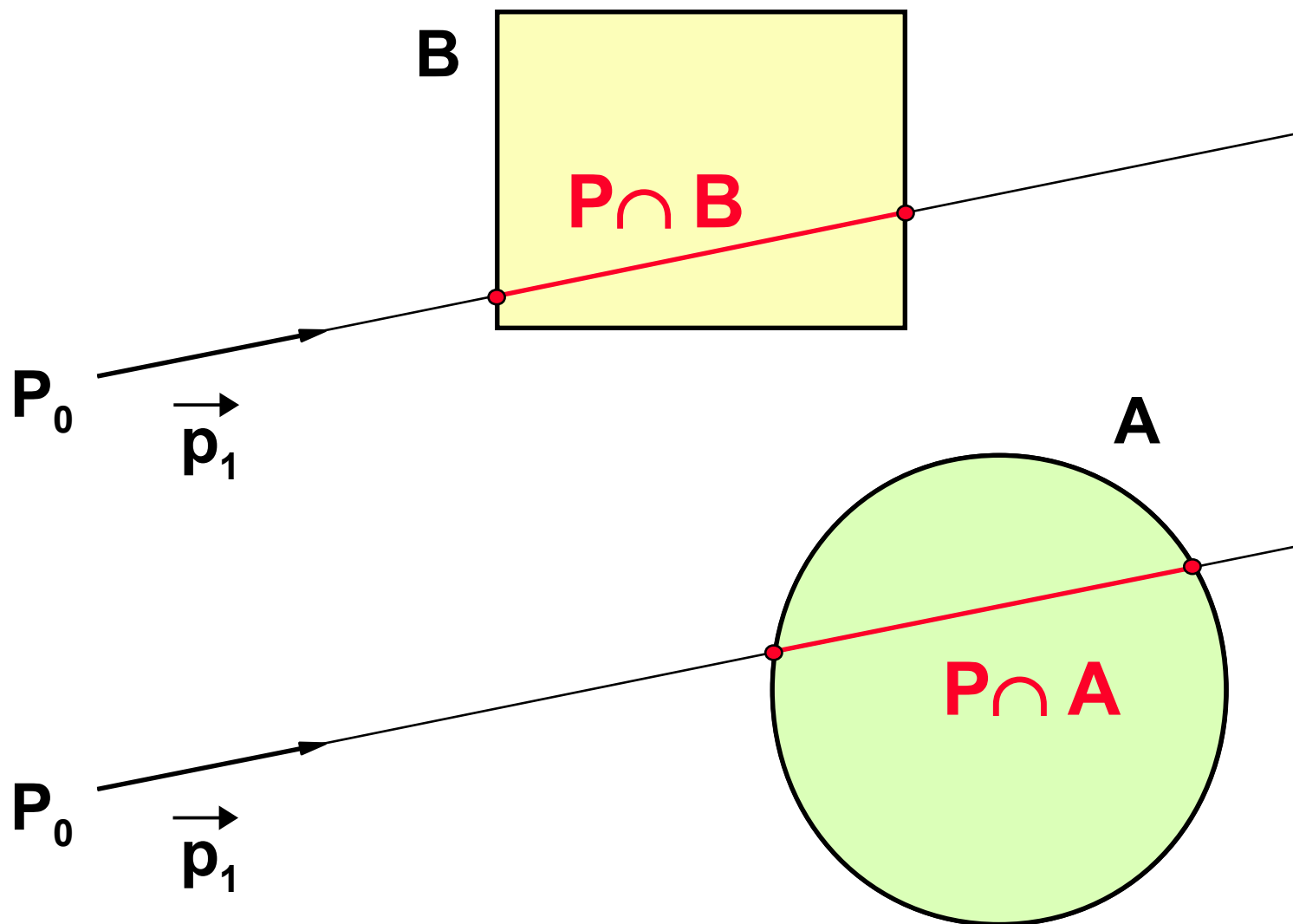


CSG representation

- ♦ **primitive solids** are easy
 - convex objects – only two intersections
- ♦ **set operations** are performed in the **1D ray-space**:
 - distributivity: $P \cap (A-B) = (P \cap A) - (P \cap B)$
 - general ray-scene intersection is a collection of line segments (intervals in 1D ray-space)
- ♦ **geometric transformations**:
 - inverse transformation applied to a ray

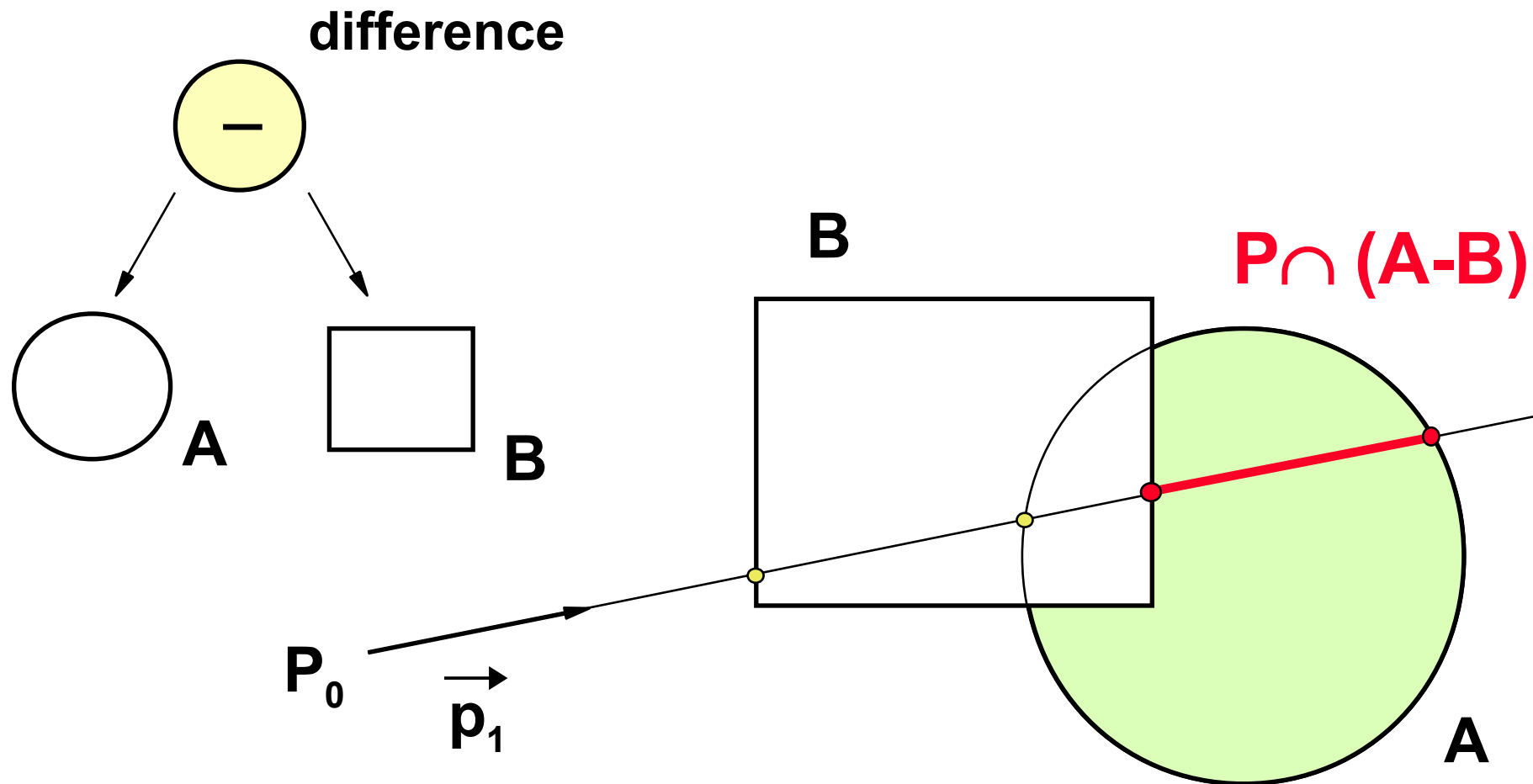


Intersections $P \cap A$, $P \cap B$





Intersection $P \cap (A-B)$





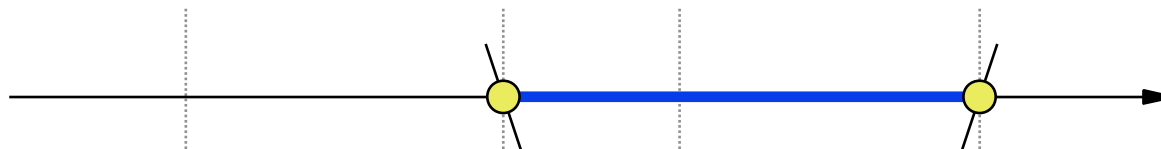
Implementation

- **ray:**
 - origin \mathbf{P}_0 and direction $\vec{\mathbf{p}}_1$
 - transforms with inverse matrices \mathbf{T}_i^{-1} (could not be efficient enough ... 1 transformation: **15+**, **18***)
- **ray vs. scene intersection** (partial & final):
 - ordered list of \mathbf{t} parameter in ray-space: $[\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \dots]$
- **set operation:**
 - generalized merging of ordered lists $[\mathbf{t}_i]$



Set operations on the ray

A



B



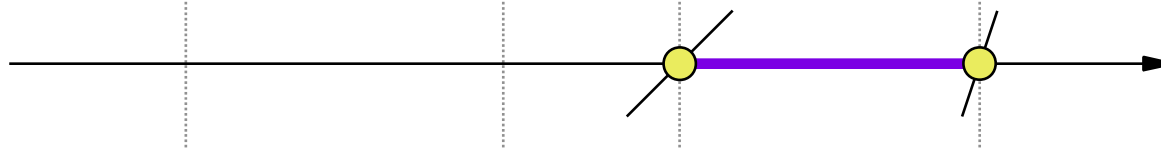
$A \cap B$



$A \cup B$



$A - B$





References

- **A. Glassner: *An Introduction to Ray Tracing*, Academic Press, London 1989, 35-119**
- **J. Foley, A. van Dam, S. Feiner, J. Hughes: *Computer Graphics, Principles and Practice*, 712-714**