Lagrange & Newton interpolation

In this section, we shall study the polynomial interpolation in the form of Lagrange and Newton. Given a sequence of (n +1) data points and a function f, the aim is to determine an n-th degree polynomial which interpolates f at these points. We shall resort to the notion of divided differences.

Interpolation

Given (n+1) points $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$, the points defined by $(x_i)_{0\le i\le n}$ are called **points of interpolation**. The points defined by $(y_i)_{0\le i\le n}$ are the **values of interpolation**. To interpolate a function f, the values of interpolation are defined as follows:

$$y_i = f(x_i), \quad \forall i = 0, \ldots, n.$$

Lagrange interpolation polynomial

The purpose here is to determine the unique polynomial of degree n, P_n which verifies

$$P_n(x_i) = f(x_i), \quad \forall i = 0, \ldots, n.$$

The polynomial which meets this equality is Lagrange interpolation polynomial

$$P_n(x)=\sum_{j=0}^{n} l_j(x) f(x_j)$$

where the l_j 's are polynomials of degree n forming a basis of P_n

$$l_j(x)=\prod_{i=0, i\neq j}^{n} \frac{x-x_i}{x_j-x_i}=\frac{x-x_0}{x_j-x_0}\cdots\frac{x-x_{j-1}}{x_j-x_{j-1}}\frac{x-x_{j+1}}{x_j-x_{j+1}}\cdots\frac{x-x_n}{x_j-x_n}$$

Properties of Lagrange interpolation polynomial and Lagrange basis

They are the l_j polynomials which verify the following property:

$$l_j(x_i)=\delta_{ji}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}, \quad \forall i=0,\ldots,n.$$

They form a basis of the vector space P_n of polynomials of degree at most equal to n

$$\sum_{j=0}^{n} \alpha_j l_j(x)=0$$

By setting: $x = x_i$, we obtain:

$$\sum_{j=0}^{n} \alpha_j l_j(x_i)=\sum_{j=0}^{n} \alpha_j \delta_{ji}=0 \;\Rightarrow \alpha_i=0$$

The set $(l_j)_{0\le j\le n}$ is linearly independent and consists of n + 1 vectors. It is thus a basis of P_n.
Finally, we can easily see that:

$$P_n(x_i)=\sum_{j=0}^{n} l_j(x_i) f(x_i)=\sum_{j=0}^{n} \delta_{ji} f(x_i)=f(x_i)$$

Example: computing Lagrange interpolation polynomials

Given a set of three data points {(0, 1), (2, 5), (4, 17)}, we shall determine the Lagrange interpolation polynomial of degree 2 which passes through these points.

First, we compute l_0, l_1 and l_2:

$$l_0(x)=\frac{(x-2)(x-4)}{8},\ l_1(x)=-\frac{x(x-4)}{4},\ l_2(x)=\frac{x(x-2)}{8}$$

Lagrange interpolation polynomial is:

$$P_n = l_0(x) + 5l_1(x) + 17l_2(x) = 1 + x^2$$

Scilab: computing Lagrange interpolation polynomial

The Scilab function `lagrange.sci` determines Lagrange interpolation polynomial. X encompasses the points of interpolation and Y the values of interpolation. P is the Lagrange interpolation polynomial.

`lagrange.sci`

```
function[P]=lagrange(X,Y) //X nodes,Y values;P is the numerical Lagrange
polynomial interpolation
n=length(X); // n is the number of nodes. (n-1) is the degree
x=poly(0,"x");P=0;
for i=1:n, L=1;
  for j=[1:i-1,i+1:n] L=L*(x-X(j))/(X(i)-X(j));end
  P=P+L*Y(i);
end
endfunction

-->X=[0;2;4]; Y=[1;5;17]; P=lagrange(X,Y)
 P = 1 + x^2
```

Such polynomials are not convenient, since numerically, it is difficult to deduce l_{j+1} from l_j. For this reason, we introduce Newton's interpolation polynomial.

Newton's interpolation polynomial and Newton's basis properties

The polynomials of Newton's basis, e_j, are defined by:

$$e_j(x)=\prod_{i=0}^{j-1}(x-x_i)=(x-x_0)(x-x_1)\cdots(x-x_{j-1}),\quad j=1,\ldots,n.$$

with the following convention:

$$e_0=1$$

Moreover

$$\begin{aligned} e_1 &= (x-x_0)\\ e_2 &= (x-x_0)(x-x_1)\\ e_3 &= (x-x_0)(x-x_1)(x-x_2)\\ &\vdots\\ e_n &= (x-x_0)(x-x_1)\cdots(x-x_{n-1}) \end{aligned}$$

The set of polynomials $(e_j)_{0\le j\le n}$ (Newton's basis) are a basis of P_n, the space of polynomials of degree at most equal to n. Indeed, they constitute an echelon-degree set of $(n + 1)$ polynomials.

Newton's interpolation polynomial of degree n related to the subdivision $\{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\}$ is:

$$P_n(x)=\sum_{j=0}^{n} \alpha_j e_j(x)=\alpha_0+\alpha_1(x-x_0)+\alpha_2(x-x_0)(x-x_1)+\ldots+\alpha_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

where

$$P_n(x_i) = f(x_i), \quad \forall i = 0, \ldots, n.$$

We shall see how to determine the coefficients $(\alpha_j)_{0\leq j\leq n}$ in the following section entitled the **divided differences**.

Divided differences

Newton's interpolation polynomial of degree n, $P_n(x)$, evaluated at x_0, gives:

$$P_n(x_0)=\sum_{j=0}^{n} \alpha_j e_j(x_0)=\alpha_0=f(x_0)=f[x_0]$$

Generally speaking, we write:

$$f[x_i] = f(x_i), \quad \forall i = 0, \ldots, n$$

$f[x_0]$ is called a zero-order **divided difference**.

Newton's interpolation polynomial of degree n, $P_n(x)$, evaluated at x_1, gives:

$$P_n(x_1)=\sum_{j=0}^{n} \alpha_j e_j(x_1)=\alpha_0+\alpha_1(x_1-x_0)=f[x_0]+\alpha_1(x_1-x_0)=f[x_1]$$

Hence

$$\alpha_1=\frac{f[x_1]-f[x_0]}{x_1-x_0}=f[x_0,x_1]$$

$f[x_1,x_0]$ is called 1^{st} **-order divided differenc**e.

Newton's interpolation polynomial of degree n, $P_n(x)$, evaluated at x_2, gives:

$$\begin{aligned}
P_n(x_2) &= \sum_{j=0}^{n} \alpha_j e_j(x_2) \\
&= \alpha_0+\alpha_1(x_2-x_0)+\alpha_2(x_2-x_0)(x_2-x_1) \\
&= f[x_0]+f[x_0,x_1](x_2-x_0)+\alpha_2(x_2-x_0)(x_2-x_1) \\
&= f[x_2]
\end{aligned}$$

Therefore:

$$\begin{aligned}
\alpha_2(x_2-x_0)(x_2-x_1) &= f[x_2]-f[x_0]-f[x_0,x_1](x_2-x_0) \\
\alpha_2 &= \frac{f[x_2]-f[x_0]-f[x_0,x_1](x_2-x_0)}{(x_2-x_0)(x_2-x_1)} \\
\alpha_2 &= \frac{f[x_2]-f[x_0]}{(x_2-x_0)(x_2-x_1)}-\frac{f[x_0,x_1]}{x_2-x_1} \\
\alpha_2 &= \frac{f[x_0,x_2]-f[x_0,x_1]}{x_2-x_1}
\end{aligned}$$

The following form is generally preferred:

$$\begin{aligned}
\alpha_2(x_2-x_0)(x_2-x_1) &= f[x_2]-f[x_0]-f[x_0,x_1](x_2-x_0) \\
\alpha_2(x_2-x_0)(x_2-x_1) &= f[x_2]-f[x_0]-f[x_0,x_1](x_2-x_0)-f[x_1]+f[x_1] \\
\alpha_2(x_2-x_0)(x_2-x_1) &= f[x_2]-f[x_1]+f[x_1]-f[x_0]-f[x_0,x_1](x_2-x_0) \\
\alpha_2(x_2-x_0)(x_2-x_1) &= f[x_2]-f[x_1]+(x_1-x_0)f[x_0,x_1]-f[x_0,x_1](x_2-x_0) \\
\alpha_2(x_2-x_0)(x_2-x_1) &= f[x_2]-f[x_1]+(x_1-x_2)f[x_0,x_1] \\
\alpha_2(x_2-x_0) &= \frac{f[x_2]-f[x_1]}{x_2-x_1}-f[x_0,x_1] \\
\alpha_2(x_2-x_0) &= f[x_1,x_2]-f[x_0,x_1]
\end{aligned}$$

Hence

$$\alpha_2=\frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}=f[x_0,x_1,x_2]$$

$f[x_0, x_1, x_2]$ is called 2nd**-order divided difference**. By recurrence, we obtain:

$$\alpha_k=\frac{f[x_1,\ldots,x_k]-f[x_0,\ldots,x_{k-1}]}{x_k-x_0}=f[x_0,\ldots,x_k]$$

$f[x_0, \ldots, x_k]$ is thus called a k^{th}**-order divided difference**. In practice, when we want to determine the 3rd-order divided difference $f[x_0, x_1, x_2, x_3]$ for instance, we need the following quantities

$$\begin{array}{lllll}
x_0 & f[x_0] & & & \\
x_1 & f[x_1] & f[x_0,x_1] & & \\
x_2 & f[x_2] & f[x_1,x_2] & f[x_0,x_1,x_2] & \\
x_3 & f[x_3] & f[x_2,x_3] & f[x_1,x_2,x_3] & f[x_0,x_1,x_2,x_3]
\end{array}$$

Hence

$$f[x_0,x_1,x_2,x_3]=\frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$$

Properties. Let $E = \{0, 1, \ldots, n\}$ and σ be a permutation of $\mathcal{G}(E)$. Then

$$f[x_{\sigma(0)}, \ldots, x_{\sigma(n)}] = f[x_0, \ldots, x_n]$$

Newton’s interpolation polynomial of degree *n*

Newton’s interpolation polynomial of degree *n* is obtained via the successive divided differences:

$$P_n(x)=f[x_0]+\sum_{j=1}^{n} f[x_0,\ldots,x_j]e_j(x)$$

An example of computing Newton’s interpolation polynomial

Given a set of 3 data points {(0, 1), (2, 5), (4, 17)}, we shall determine Newton’s interpolation polynomial of degree 2 which passes through these points.

$$\begin{array}{llll}
x_0=0 & f[x_0]=1 & & \\
x_1=2 & f[x_1]=5 & f[x_0,x_1]=\dfrac{5-1}{2-0}=2 & \\
x_2=4 & f[x_2]=17 & f[x_1,x_2]=\dfrac{17-5}{4-2}=6 & f[x_0,x_1,x_2]=\dfrac{6-2}{4-0}=1
\end{array}$$

Consequently:

$$P_2(x) = f[x_0] + f[x_0, x_1]x + f[x_0, x_1, x_2]x(x - 2) = 1 + 2x + x(x - 2) = 1 + x^2$$

Scilab: computing Newton's interpolation polynomial

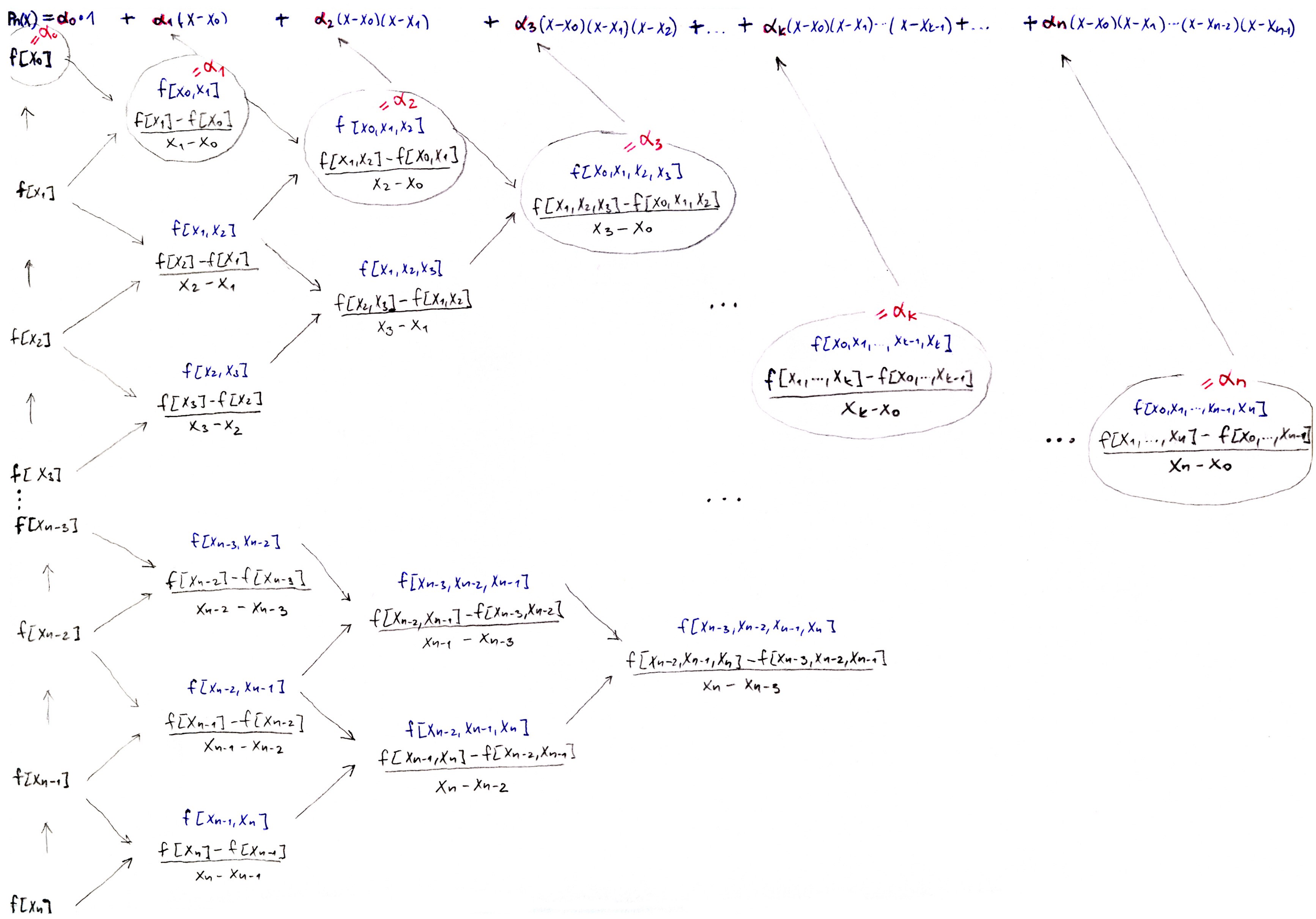
Scilab function `newton.sci` determines Newton's interpolation polynomial. *X* contains the points of interpolation and *Y* the values of interpolation. *P* is Newton's interpolation polynomial computed by means of divided differences.

newton.sci

```
function[P]=newton(X,Y) //X nodes,Y values;P is the numerical
Newton polynomial
n=length(X); // n is the number of nodes. (n-1) is the degree
for j=2:n,
  for i=1:n-j+1,Y(i,j)=(Y(i+1,j-1)-Y(i,j-1))/(X(i+j-1)-X(i));end,
end,
x=poly(0,"x");
P=Y(1,n);
for i=2:n, P=P*(x-X(i))+Y(i,n-i+1); end
endfunction;
```

Therefore, we obtain:

```
-->X=[0;2;4]; Y=[1;5;17]; P=newton(X,Y)
 P = 1 + x^2
```



Pn(x) = α0·1 + α1(x-x0) + α2(x-x0)(x-x1) + α3(x-x0)(x-x1)(x-x2) + … + αk(x-x0)(x-x1)···(x-xk-1) + … + αn(x-x0)(x-x1)···(x-xn-2)(x-xn-1)
f[x0] = α0
f[x1]
f[x2]
f[x3]
f[xn-3]
f[xn-2]
f[xn-1]
f[xn]
f[x0,x1] = α1
(f[x1] - f[x0]) / (x1 - x0)
f[x1,x2]
(f[x2] - f[x1]) / (x2 - x1)
f[x2,x3]
(f[x3] - f[x2]) / (x3 - x2)
f[x0,x1,x2] = α2
(f[x1,x2] - f[x0,x1]) / (x2 - x0)
f[x1,x2,x3]
(f[x2,x3] - f[x1,x2]) / (x3 - x1)
f[x0,x1,x2,x3] = α3
(f[x1,x2,x3] - f[x0,x1,x2]) / (x3 - x0)
f[x0,x1,…,xk-1,xk] = αk
(f[x1,…,xk] - f[x0,…,xk-1]) / (xk - x0)
f[x0,x1,…,xn-1,xn] = αn
(f[x1,…,xn] - f[x0,…,xn-1]) / (xn - x0)
f[xn-3,xn-2]
(f[xn-2] - f[xn-3]) / (xn-2 - xn-3)
f[xn-2,xn-1]
(f[xn-1] - f[xn-2]) / (xn-1 - xn-2)
f[xn-1,xn]
(f[xn] - f[xn-1]) / (xn - xn-1)
f[xn-3,xn-2,xn-1]
(f[xn-2,xn-1] - f[xn-3,xn-2]) / (xn-1 - xn-3)
f[xn-2,xn-1,xn]
(f[xn-1,xn] - f[xn-2,xn-1]) / (xn - xn-2)
f[xn-3,xn-2,xn-1,xn]
(f[xn-2,xn-1,xn] - f[xn-3,xn-2,xn-1]) / (xn - xn-3)