$$\omega = [x, y, z],\ \ x^2 + y^2 + z^2 = 1$$
$$\omega = [\theta, \phi];\ \ \theta \in [0, \pi];\ \ \phi \in [0, 2\pi]$$

$$\theta = \arccos z$$
$$\phi = \arctan \frac{y}{x}$$
$$x = \sin\theta \cos\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\theta$$

$$\mathrm{d}\omega = \mathrm{d}A\, \frac{\cos\theta}{r^2}$$
$$\mathrm{d}\omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$$

$$\Phi(S, t) = \frac{\mathrm{d}Q(S, t)}{\mathrm{d}t}$$
$$E(\vec{x}) = B(\vec{x}) = \frac{\mathrm{d}\Phi(\vec{x})}{\mathrm{d}A_{\vec{x}}}$$
$$I(\omega) = \frac{\mathrm{d}\Phi(\omega)}{\mathrm{d}\omega}$$
$$L(\vec{x}, \omega) = \frac{\mathrm{d}^2\Phi(\vec{x}, \omega)}{\mathrm{d}A_{\vec{x}}\, \mathrm{d}\omega \cos\theta} = \frac{\mathrm{d}I(\omega)}{\mathrm{d}A_{\vec{x}} \cos\theta} = \frac{\mathrm{d}E(\vec{x})}{\mathrm{d}\omega \cos\theta}$$
$$E(\vec{x}) = \int_\Omega L(\vec{x}, \omega) \cos\theta\, \mathrm{d}\omega$$
$$I(\omega) = \int_{A_{\vec{x}}} L(\vec{x}, \omega) \cos\theta\, \mathrm{d}A_{\vec{x}}$$
$$\Phi(\vec{x}) = \int_{A_{\vec{x}}} E(\vec{x})\, \mathrm{d}A_{\vec{x}} = \int_{A_{\vec{x}}} \int_\Omega L(\vec{x}, \omega) \cos\theta\, \mathrm{d}\omega\, \mathrm{d}A_{\vec{x}}$$

$$I(\omega) = I_0 \cos\measuredangle(\omega, \vec{d}) = I_0(\omega \cdot \vec{d})$$
$$I(\omega) = \begin{cases} I_0 & \measuredangle(\omega, \vec{d}) < \tau \\ 0 & \text{otherwise} \end{cases}$$

$$f_r(\omega_i \to \omega_o) = \frac{\mathrm{d}L_o(\omega_o)}{\mathrm{d}E(\omega_i)} = \frac{\mathrm{d}L_o(\omega_o)}{L_i(\omega_i) \cos\theta_i\, \mathrm{d}\omega_i}$$
$$f_r(\theta_i, \phi_i; \theta_o, \phi_o) = f_r(\theta_i, \phi_i + \phi; \theta_o, \phi_o + \phi) = f_r(\theta_i, \theta_o; \phi_o - \phi_i)$$
$$L_o(\omega_o) = \int_\Omega f_r(\omega_i \to \omega_o) L_i(\omega_i) \cos\theta_i\, \mathrm{d}\omega_i$$

$$\frac{B}{E} = \frac{\int L_o(\omega_o) \cos\theta_o\, \mathrm{d}\omega_o}{\int L_i(\omega_i) \cos\theta_i\, \mathrm{d}\omega_i} =$$
$$= \frac{\int \left(\int f_r(\omega_i \to \omega_o) L_i(\omega_i) \cos\theta_i\, \mathrm{d}\omega_i\right) \cos\theta_o\, \mathrm{d}\omega_o}{\int L_i(\omega_i) \cos\theta_i\, \mathrm{d}\omega_i} \leq 1$$

$$\rho(\omega_o) = \int_\Omega f_r(\omega_i \to \omega_o) \cos\theta_i\, \mathrm{d}\omega_i$$

$$f_r(\omega_i \to \omega_o) = f_{r,d}$$

$$L_o(\omega_o) = f_{r,d} \int_\Omega L_i(\omega_i) \cos\theta_i \, \mathrm{d}\omega_i = f_{r,d} \; E$$

$$\rho_d = \pi \; f_{r,d}$$

$$\omega_o = \omega_r = 2(\omega_i \cdot \vec{n})\vec{n} - \omega_i$$

$$L_o(\omega_o) = F(\omega_i) L_i(\omega_i) = F(\omega_o) L_i(\omega_i)$$

$$f_r(\omega_i \to \omega_o) = F(\omega_i) \frac{\delta(\omega_i - \omega_o)}{\cos\theta_i} = F(\omega_i) \frac{\delta(\cos\theta_i - \cos\theta_o)\delta(\phi_i - \phi_o \pm \pi)}{\cos\theta_i}$$

$$\eta_i \sin\theta_i = \eta_o \sin\theta_o$$

$$\eta_{io} = \frac{\eta_i}{\eta_o}$$

$$\omega_o = -\eta_{io}\omega_i - \left(\eta_{io}\cos\theta_i + \sqrt{1 - \eta_{io}^2(1 - \cos^2\theta_i)}\right)\vec{n}$$

$$\theta_i = \arcsin\left(\frac{\eta_o}{\eta_i}\right)$$

$$L_o = L_i \left(\frac{\eta_o}{\eta_i}\right)^2$$

$$f_t(\omega_i \to \omega_o) = \left(\frac{\eta_o}{\eta_i}\right)^2 (1 - F(\omega_i)) \frac{\delta(\omega_i - \omega_o)}{\cos\theta_i} = \dots$$

$$F_s = \left|\frac{\eta_i \cos\theta_i - \eta_o \cos\theta_o}{\eta_i \cos\theta_i + \eta_o \cos\theta_o}\right|^2$$

$$F_p = \left|\frac{\eta_i \cos\theta_o - \eta_o \cos\theta_i}{\eta_i \cos\theta_o + \eta_o \cos\theta_i}\right|^2$$

$$F = \frac{F_s + F_p}{2}$$

$$L_o(\omega_o) = L_i(\omega_i)\,(k_d \cos\theta_i + k_s \cos^n\theta_r)$$

$$\cos\theta_i = \omega_i \cdot \vec{n}; \;\; \cos\theta_r = \omega_o \cdot \omega_r$$

$$f_r(\omega_i \to \omega_o) = k_d + k_s \frac{\cos^n\theta_r}{\cos\theta_i}$$

$$f_r^{modif}(\omega_i \to \omega_o) = \frac{\rho_d}{\pi} + \frac{\rho_s}{2\pi}(n+2)\cos^n\theta_r; \;\; \rho_d + \rho_s \leq 1$$

$$\omega_h = \frac{\omega_i + \omega_o}{||\omega_i + \omega_o||}$$

$$f_r(\omega_i \to \omega_o) = \frac{F(\theta_i) G(\omega_i, \omega_o) D(\omega_h)}{4 \cos\theta_i \cos\theta_o}$$

$$D(\omega_h) = e^{-\frac{\theta_h^2}{m^2}}; \;\; \theta_h = \measuredangle(\omega_h, \vec{n})$$

$$D(\omega_h) = \frac{1}{\pi m^2}(\omega_h \cdot \vec{n})^{\frac{2}{m^2}-2}$$

$$D(\omega_h) = \frac{1}{\pi m^2}\frac{1}{(\omega_h \cdot \vec{n})^4} e^{\frac{(\omega_h \cdot \vec{n})^2 - 1}{m^2(\omega_h \cdot \vec{n})^2}}$$

$$D(\omega_h) = \frac{m^2}{\pi\left((\omega_h \cdot \vec{n})^2(m^2-1)+1\right)^2}$$

$$G(\omega_i, \omega_o) = \min\left\{1, \frac{2(\omega_h \cdot \vec{n})(\omega_o \cdot \vec{n})}{(\omega_o \cdot \omega_h)}, \frac{2(\omega_h \cdot \vec{n})(\omega_i \cdot \vec{n})}{(\omega_o \cdot \omega_h)}\right\}$$

$$G(\omega_i, \omega_o) = G(\omega_i)G(\omega_o)$$

$$G(\omega_i) = \frac{2(\omega_i \cdot \vec{n})}{(\omega_i \cdot \vec{n}) + \sqrt{m^2 + (1-m^2)(\omega_i \cdot \vec{n})^2}}$$

$$G(\omega_o) = \frac{2(\omega_o \cdot \vec{n})}{(\omega_o \cdot \vec{n}) + \sqrt{m^2 + (1-m^2)(\omega_o \cdot \vec{n})^2}}$$

$$F(\theta_i) = F(0) + (1 - F(0))(1 - \cos\theta_i)^5;\ F(0) = \left\|\frac{\eta_i - \eta_o}{\eta_i + \eta_o}\right\|^2$$

$$f_r(\omega_i \to \omega_o) = \frac{\rho}{\pi}(A + B\max(0, \cos(\theta_i - \theta_o))\sin\alpha\tan\beta)$$

$$A = 1 - \frac{\sigma^2}{2(\sigma^2 + 0.33)};\ B = \frac{0.45\sigma^2}{\sigma^2 + 0.09};\ \alpha = \max(\theta_i, \theta_o);\ \beta = \min(\theta_i, \theta_o)$$

$$\vec{N} = \frac{\sum_{i=1}^{n} \vec{N}_i}{\left\|\sum_{i=1}^{n} \vec{N}_i\right\|}$$

$$I_X = (1-v)\left[(1-u)\cdot I_A + u \cdot I_B\right] + v\left[(1-u)\cdot I_C + u\cdot I_D\right]$$

$$I_{AB} = I_A\frac{y - B_y}{A_y - B_y} + I_B\frac{A_y - y}{A_y - B_y};\ A = [A_x, A_y];\ B = [B_x, B_y]$$

$$I_{AC} = I_A\frac{y - C_y}{A_y - C_y} + I_C\frac{A_y - y}{A_y - C_y};\ A = [A_x, A_y];\ C = [C_x, C_y]$$

$$I_X = I_{AB}\frac{(AC)_x - x}{(AC)_x - (AB)_x} + I_{AC}\frac{x - (AB)_x}{(AC)_x - (AB)_x};\ X = [x, y]$$

$$\vec{N}_X = (1-v)\left[(1-u)\cdot \vec{N}_A + u\cdot \vec{N}_B\right] + v\left[(1-u)\cdot \vec{N}_C + u \cdot \vec{N}_D\right]$$

$$\vec{N}_{AB} = \vec{N}_A\frac{y - B_y}{A_y - B_y} + \vec{N}_B\frac{A_y - y}{A_y - B_y}$$

$$\vec{N}_{AC} = \vec{N}_A\frac{y - C_y}{A_y - C_y} + \vec{N}_C\frac{A_y - y}{A_y - C_y}$$

$$\vec{N}_X = \vec{N}_{AB}\frac{(AC)_x - x}{(AC)_x - (AB)_x} + \vec{N}_{AC}\frac{x - (AB)_x}{(AC)_x - (AB)_x}$$

$$I = (I^R, I^G, I^B)$$

$$
\begin{aligned}
I^\lambda &= I_a^\lambda k_d^\lambda + \sum_{i=1}^{m} f_i^{att} I_i^\lambda \left[k_d^\lambda \cos\theta_i + k_s^\lambda \cos^n \theta_r\right] = \\
&= I_a^\lambda k_d^\lambda + \sum_{i=1}^{m} f_i^{att} I_i^\lambda \left[k_d^\lambda (\vec{N} \cdot \vec{L}_i) + k_s^\lambda (\vec{R}_i \cdot \vec{V})^n\right] = \\
&\approx I_a^\lambda k_d^\lambda + \sum_{i=1}^{m} f_i^{att} I_i^\lambda \left[k_d^\lambda (\vec{N} \cdot \vec{L}_i) + k_s^\lambda (\vec{N} \cdot \vec{H}_i)^n\right] \\
&\text{for } \lambda = R, G, B
\end{aligned}
$$

$$
\begin{aligned}
&p : P(t) = P_0 + t \cdot \vec{p_1};\ t \in [0,1] \\
&\alpha : x \cdot n_x + y \cdot n_y + z \cdot n_z + D = 0;\ \vec{n} = [n_x, n_y, n_z] \\
&p \cap \alpha : t = -(\vec{n} \cdot P_0 + D)/(\vec{n} \cdot \vec{p_1})
\end{aligned}
$$

$$
\alpha(u,v) = P + u \cdot \vec{u} + v \cdot \vec{v};\ \vec{u} = [u_x, u_y, u_z];\ \vec{v} = [v_x, v_y, v_z];\ \vec{n} = \vec{u} \times \vec{v}
$$

$$
\begin{aligned}
u \cdot u_x + v \cdot \vec{v}_x &= x - P_x \\
u \cdot u_y + v \cdot \vec{v}_y &= y - P_y \\
u = ?,\ v = ?
\end{aligned}
$$

$$
0 \le u,\ v \le 1
$$

$$
0 \le u,\ v,\ u + v \le 1
$$

$$
\begin{aligned}
&p : P(t) = P_0 + t \cdot \vec{p_1};\ t \in [0,1] \\
&t_0 = (\vec{v} \cdot \vec{p_1});\ \vec{v} = S - P_0 \\
&D^2 = (\vec{v} \cdot \vec{v}) - t_0^2 \\
&t^2 = R^2 - D^2 \\
&X = \begin{cases} P(t_0) & \text{if } t^2 = 0 \\ P(t_0 + t) \cup P(t_0 - t) & \text{if } t^2 > 0 \\ \emptyset & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
K \cdot B + \sum_{i=1}^{K} p_i I_i \overset{?}{<} \sum_{i=1}^{K} I_i
$$

$$
\cos\alpha \ge \sqrt{1 - \frac{R_1 + R_2}{\|S_1 - S_2\|}}
$$

$$
\Pr(X \in D) = \int_D p(x)\,\mathrm{d}x
$$

$$
\Pr(a < X \le b) = \int_a^b p(t)\,\mathrm{d}t
$$

$$
P(x) \equiv \Pr(X \le x) = \int_{-\infty}^{x} p(t)\,\mathrm{d}t
$$

$$E[X] = \int_D \vec{x}\, p(\vec{x})\, \mathrm{d}\vec{x}$$
$$V[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$$
$$V\left[\sum_i X_i\right] = \sum_i V[X_i];\; V[aX] = a^2 V[X]$$
$$\text{if } Y = f(X) \text{ then } E[Y] = \int_D f(\vec{x}) p(\vec{x})\, \mathrm{d}\vec{x}$$

$$\text{if } I = \int f(\vec{x})\, \mathrm{d}\vec{x} \;\text{ then } \langle I \rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{f(\xi_i)}{p(\xi_i)} \text{ and } E[\langle I \rangle] = I \text{ for } \xi_i \propto p(\vec{x})$$
$$F_{prim} = \frac{f(X)}{p(X)}$$
$$V[F_{prim}] = \sigma^2_{prim} = E[F^2_{prim}] - E[F_{prim}]^2 = \int \frac{f(\vec{x})^2}{p(\vec{x})}\, \mathrm{d}\vec{x} - I^2$$
$$\text{If } F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)} \text{ then } E[F_N] = E\left[\frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right] = \frac{1}{N} E\left[\sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right] = \frac{1}{N} \cdot N \cdot I = I$$
$$V[F_N] = V\left[\frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right] = \frac{1}{N^2} N \cdot V\left[\frac{f(X_i)}{p(X_i)}\right] = \frac{1}{N} V[F_{prim}]$$

$$\beta = Q - E[F_N]$$
$$\Pr\left\{\lim_{N\to\infty} F_N(X_1, \ldots X_N) = Q\right\} = 1$$
$$\lim_{N\to\infty} \beta[F_N] = \lim_{N\to\infty} V[F_N] = 0$$

$$E(\vec{x}) = \int_\Omega L(\vec{x}, \omega_i) \cos\theta_i\, \mathrm{d}\omega_i$$
$$p(\omega) = \frac{1}{2\pi};\; F_N = \frac{1}{N} \sum_{k=1}^{N} \frac{L(\vec{x}, \omega_{i,k}) \cos\theta_{i,k}}{p(\omega_{i,k})} = \frac{2\pi}{N} \sum_{i=1}^{N} L_i(\vec{x}, \omega_{i,k}) \cos\theta_{i,k}$$
$$p(\omega) = \frac{\cos\theta}{\pi};\; F_N = \frac{1}{N} \sum_{k=1}^{N} \frac{L(\vec{x}, \omega_{i,k}) \cos\theta_{i,k}}{p(\omega_{i,k})} = \frac{\pi}{N} \sum_{i=1}^{N} L_i(\vec{x}, \omega_{i,k})$$

$$E(\vec{x}) = \int_{A_{\vec{x}}} L(\vec{y} \to \vec{x}) V(\vec{y} \leftrightarrow \vec{x}) \frac{\cos\theta_y \cos\theta_x}{\|\vec{y} - \vec{x}\|^2}\, \mathrm{d}A_{\vec{x}}$$
$$p(\vec{y}) = \frac{1}{|A_{\vec{x}}|};\; F_N = \frac{|A_{\vec{x}}|}{N} \sum_{k=1}^{N} L(\vec{y}_k \to \vec{x}) V(\vec{y}_k \leftrightarrow \vec{x}) \frac{\cos\theta_{y,k} \cos\theta_x}{\|\vec{y}_k - \vec{x}\|^2}$$

$$L_o(\vec{x}, \omega_o) = \int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) L(\vec{x}, \omega_i) \cos\theta_i\, \mathrm{d}\omega_i$$
$$p(\omega_i);\; F_N = \frac{1}{N} \sum_{k=1}^{N} \frac{f_r(\vec{x}, \omega_{i,k} \to \omega_o) L(\vec{x}, \omega_{i,k}) \cos\theta_{i,k}}{p(\omega_{i,k})}$$

$$L_o(\vec{x},\omega_o) = \int_{A_{\vec{x}}} f_r(\vec{y}\to\vec{x}\to\omega_o)L(\vec{y}\to\vec{x})V(\vec{y}\leftrightarrow\vec{x})\frac{\cos\theta_y\cos\theta_x}{\|\vec{y}-\vec{x}\|^2}\,\mathrm{d}A_{\vec{x}}$$

$$p(\vec{y}) = \frac{1}{|A_{\vec{x}}|};\ F_N = \frac{|A_{\vec{x}}|}{N}\sum_{k=1}^{N} f_r(\vec{y}\to\vec{x}\to\vec{z})L(\vec{y}_k\to\vec{x})V(\vec{y}_k\leftrightarrow\vec{x})\frac{\cos\theta_{y,k}\cos\theta_x}{\|\vec{y}_k-\vec{x}\|^2}$$

$$p(\omega) = \frac{p(\theta,\phi)}{\sin\theta} \text{ while } \int_\Omega p(\omega)\,\mathrm{d}\omega = 1$$

$$p(\theta,\phi) = p(\theta)p(\phi|\theta) \text{ where } p(\theta) = \int_0^{2\pi} p(\theta,\phi)\,\mathrm{d}\phi \text{ and } p(\phi|\theta) = \frac{p(\theta,\phi)}{p(\theta)}$$

$$P(\theta) = \int_0^\theta p(\theta')\,\mathrm{d}\theta'\,;\ P(\phi|\theta) = \int_0^\phi p(\phi'|\theta)\,\mathrm{d}\phi'$$

$\theta = P^{-1}(\xi_1),\ \phi = P^{-1}(\xi_2|\xi_1)$ where $\xi_1,\xi_2 \in$ unif(0,1) and $P(\theta), P(\phi|\theta)$ are **cdf**

If $r_1, r_2 \in$ unif(0,1) and $p(\omega) = \dfrac{\cos\theta}{\pi}$ then

$\theta_i = \measuredangle(\omega_i,\vec{n});\ \theta_r = \measuredangle(\omega_r,\omega_o)$

$f_r^{modif}(\omega_i\to\omega_o) = \dfrac{\rho_d}{\pi} + \dfrac{\rho_s}{2\pi}(n+2)\cos^n\theta_r;\ \rho_d+\rho_s \le 1$

$\theta = \arccos(\sqrt{r_2})$

$\phi = 2\pi r_1$

$x = \cos(2\pi r_1)\sqrt{1-r_2}$

$y = \sin(2\pi r_1)\sqrt{1-r_2}$

$z = \sqrt{r_2}$

if $r_1, r_2 \in$ unif(0,1) and $p(\omega) = \dfrac{n+1}{2\pi}\cos^n\theta$ then

$\theta = \arccos\left(r_2^{\frac{1}{n+1}}\right)$

$\phi = 2\pi r_1$

$x = \cos(2\pi r_1)\sqrt{1-r_2^{\frac{2}{n+1}}}$

$y = \sin(2\pi r_1)\sqrt{1-r_2^{\frac{2}{n+1}}}$

$z = \sqrt{r_2^{\frac{1}{n+1}}}$

$$I = \int f(\vec{x})\,\mathrm{d}\vec{x} = \int [f(\vec{x})-g(\vec{x})]\,\mathrm{d}\vec{x} + \int g(\vec{x})\,\mathrm{d}\vec{x}$$

If $I = \displaystyle\int_\Omega f(\vec{x})\,\mathrm{d}\vec{x} = \sum_{i=1}^{N}\int_{\Omega_i} f(\vec{x})\,\mathrm{d}\vec{x} = \sum_{i=1}^{N} I_i$ then $\langle I_{strat}\rangle = \dfrac{1}{N}\displaystyle\sum_{i=1}^{N} f(X_i),\ X_i\in\Omega_i$

$$F = \sum_{i=1}^{N}\frac{1}{N_i}\sum_{k=1}^{N_i} w_i(X_{i,k})\frac{f(X_{i,k})}{p_i(X_{i,k})};\ \sum_{i=1}^{N} w_i(X_{i,k}) = 1 \text{ for all } i,k$$

$$E[F] = \sum_{i=1}^{N}\frac{1}{N_i}\sum_{k=1}^{N_i}\int_{\Omega_k} w_i(\vec{x})\frac{f(\vec{x})}{p_i(\vec{x})}p_i(\vec{x})\,\mathrm{d}\vec{x} = \int_\Omega\sum_{i=1}^{N} w_i(\vec{x})f(\vec{x})\,\mathrm{d}\vec{x} = \int_\Omega f(\vec{x})\,\mathrm{d}\vec{x}$$

If $\hat{w}_i(X_{i,k}) = \dfrac{N_i p_i(X_{i,k})}{\sum_j N_j p_j(X_{i,k})}$ then $\hat{F} = \displaystyle\sum_{i=1}^{N} \frac{1}{N_i} \sum_{k=1}^{N_i} \frac{N_i p_i(X_{i,k})}{\sum_j N_j p_j(X_{i,k})} \frac{f(X_{i,k})}{p_i(X_{i,k})} = \sum_{i=1}^{N} \sum_{k=1}^{N_i} \frac{f(X_{i,k})}{\sum_j N_j p_j(X_{i,k})}$

$$L_o(\vec{x}, \omega_o) = L_e(\vec{x}, \omega_o) + \int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) L_i(\vec{x}, \omega_i) \cos\theta_i \, \mathrm{d}\omega_i \,; \;\; L_i(\vec{x}, \omega_i) = L_o(r(\vec{x}, \omega_i), -\omega_i)$$

$$L(\vec{x}, \omega_o) = L_e(\vec{x}, \omega_o) + \int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) L(r(\vec{x}, \omega_i), -\omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

$L(\vec{x}, \omega_o) = L_e(\vec{x}, \omega_o) + \displaystyle\int_\mathcal{M} f_r(\vec{y} \to \vec{x} \to \omega_o) L(\vec{y} \to \vec{x}) V(\vec{y} \leftrightarrow \vec{x}) \frac{\cos\theta_y \cos\theta_x}{\|\vec{y} - \vec{x}\|^2} \, \mathrm{d}A_{\vec{y}}$ where $\mathcal{M}$ are scene surfaces

$$f_r(\vec{y} \to \vec{x} \to \omega_o) = \frac{\rho(\vec{x})}{\pi}$$

$$L(\vec{x}, \omega_o) = L_e(\vec{x}, \omega_o) + \frac{\rho(\vec{x})}{\pi} \int_\mathcal{M} L(\vec{y} \to \vec{x}) V(\vec{y} \leftrightarrow \vec{x}) G(\vec{y} \leftrightarrow \vec{x}) \, \mathrm{d}A_{\vec{y}} \,; \;\; G(\vec{y} \leftrightarrow \vec{x}) = \frac{\cos\theta_y \cos\theta_x}{\|\vec{y} - \vec{x}\|^2}$$

$$B(\vec{x}) = B_e(\vec{x}) + \frac{\rho(\vec{x})}{\pi} \int_\mathcal{M} B(\vec{y}) V(\vec{y} \leftrightarrow \vec{x}) G(\vec{y} \leftrightarrow \vec{x}) \, \mathrm{d}A_{\vec{y}}$$

$$B(\vec{x}) = B_e(\vec{x}) + \frac{\rho(\vec{x})}{\pi} \sum_{k=1}^{N} \left(B_k(\vec{y}) \int_{A_{\vec{y},k}} V(\vec{y} \leftrightarrow \vec{x}) G(\vec{y} \leftrightarrow \vec{x}) \, \mathrm{d}A_{\vec{y},k} \right)$$

If $B_i(\vec{x}) = \dfrac{1}{|A_{\vec{x},i}|} \displaystyle\int_{A_{\vec{x},i}} B(\vec{x}) \, \mathrm{d}A_{\vec{x},i}$ then:

$$B_i(\vec{x}) = \frac{1}{|A_{\vec{x},i}|} \int_{A_{\vec{x},i}} \left(B_e(\vec{x}) + \frac{\rho(\vec{x})}{\pi} \sum_{k=1}^{N} \left(B_k(\vec{y}) \int_{A_{\vec{y},k}} V(\vec{y} \leftrightarrow \vec{x}) G(\vec{y} \leftrightarrow \vec{x}) \, \mathrm{d}A_{\vec{y},k} \right) \right) \mathrm{d}A_{\vec{x},i} =$$

$$= B_{e,i}(\vec{x}) + \frac{\rho_i(\vec{x})}{\pi} \sum_{k=1}^{N} \left(B_{i,k}(\vec{y}) \frac{1}{|A_{\vec{x},i}|} \int_{A_{\vec{x},i}} \int_{A_{\vec{y},k}} V(\vec{y} \leftrightarrow \vec{x}) G(\vec{y} \leftrightarrow \vec{x}) \, \mathrm{d}A_{\vec{y},k} \, \mathrm{d}A_{\vec{x},i} \right) =$$

$= B_{e,i}(\vec{x}) + \rho_i(\vec{x}) \displaystyle\sum_{k=1}^{N} B_{i,k}(\vec{y}) F_{i,k}(\vec{x}, \vec{y})$ where $F_{i,k}(\vec{x}, \vec{y}) = \dfrac{1}{|A_{\vec{x},i}|} \displaystyle\int_{A_{\vec{x},i}} \int_{A_{\vec{y},k}} \frac{V(\vec{y} \leftrightarrow \vec{x}) G(\vec{y} \leftrightarrow \vec{x})}{\pi} \, \mathrm{d}A_{\vec{y},k} \, \mathrm{d}A_{\vec{x},i}$

$$B_i = B_{e,i} + \rho_i \sum_{k=1}^{N} B_k \cdot F_{i \to k}$$

$$L(\vec{x}, \omega_o) = L_e(\vec{x}, \omega_o) + \int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) L(r(\vec{x}, \omega_i), -\omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

$$(T \circ L)(\vec{x}, \omega_o) \equiv \int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) L(r(\vec{x}, \omega_i), -\omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

$$L(\vec{x}, \omega_o) = L_e(\vec{x}, \omega_o) + (T \circ L)(\vec{x}, \omega_o)$$

$$L = L_e + T \circ L = L_e + (T \circ (L_e + T \circ L)) = L_e + T \circ L_e + T^2 \circ L$$

$$L = L_e + T \circ (L_e + T \circ (L_e + T \circ (L_e + \cdots = L_e + T \circ L_e + T^2 \circ L_e + T^3 \circ L_e + \ldots$$

$L = (I - T)^{-1} \circ L_e$ where $(I - T)^{-1} = I + T + T^2 + \ldots$

$$L = \sum_{i=0}^{n} (T^i \circ L_e) + T^{n+1} \circ L$$

If $\lim_{n\to\infty} T^{n+1} \circ L = 0$ then $L = \sum_{i=0}^{\infty} (T^i \circ L_e)$

$$F' = \begin{cases} \frac{F}{q} & \xi < q \quad \text{(probability } q\text{)} \\ 0 & \text{otherwise (probability } 1-q\text{)} \end{cases}$$

$E[F'] = E\left[\frac{F}{q}\right] \cdot q + 0 \cdot (1-q) = E[F]$ where F is an estimator

$q = \rho(\vec{x})$ or $q = \min\left\{1, \frac{f_r(\vec{x}, \omega_i \to \omega_o) \cos\theta_i}{p(\omega_i)}\right\}$

$p(\omega_i) \propto f_r(\vec{x}, \omega_i \to \omega_o) \cos\theta_i$

$$p(\omega_i) = \frac{f_r(\vec{x}, \omega_i \to \omega_o) \cos\theta_i}{\int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) \cos\theta_i \, \mathrm{d}\omega_i} = \frac{f_r(\vec{x}, \omega_i \to \omega_o) \cos\theta_i}{\rho(\omega_o)}$$

$$L_o(\vec{x}, \omega_o) = \int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) L_e(r(\vec{x}, \omega_i), -\omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

$$p(\omega_i);\ F_N = \frac{1}{N} \sum_{k=1}^{N} \frac{f_r(\vec{x}, \omega_{i,k} \to \omega_o) L_e(r(\vec{x}, \omega_{i,k}), -\omega_{i,k}) \cos\theta_{i,k}}{p(\omega_{i,k})}$$

$$L_o(\vec{x}, \omega_o) = \int_{A_{\vec{x}}} f_r(\vec{y} \to \vec{x} \to \omega_o) L_e(\vec{y} \to \vec{x}) V(\vec{y} \leftrightarrow \vec{x}) \frac{\cos\theta_y \cos\theta_x}{\|\vec{y} - \vec{x}\|^2} \, \mathrm{d}A_{\vec{x}}$$

$$p(\vec{y}) = \frac{1}{|A_{\vec{x}}|};\ F_N = \frac{|A_{\vec{x}}|}{N} \sum_{k=1}^{N} f_r(\vec{y} \to \vec{x} \to \vec{z}) L_e(\vec{y}_k \to \vec{x}) V(\vec{y}_k \leftrightarrow \vec{x}) \frac{\cos\theta_{y,k} \cos\theta_x}{\|\vec{y}_k - \vec{x}\|^2}$$

$$p_1(\omega) = \frac{\cos\theta_x}{\pi},\ p_2(\omega) = \frac{1}{|A_{\vec{x}}|} \frac{\|\vec{y} - \vec{x}\|^2}{\cos\theta_y};\ w_1(\omega) = \frac{p_1(\omega)}{p_1(\omega) + p_2(\omega)},\ w_2(\omega) = \frac{p_2(\omega)}{p_1(\omega) + p_2(\omega)}$$

$$F_N^{comb} = \frac{1}{2N} \sum_{k=1}^{2N} \frac{f_r(\vec{x}, \omega_k \to \omega_o) L_e(r(\vec{x}, \omega_k), -\omega_k)}{\frac{1}{\pi} + \frac{1}{|A_{\vec{x}}|} \frac{\|\vec{y} - \vec{x}\|^2}{\cos\theta_y}}$$

$$W(\vec{x}, \omega_o) = W_e(\vec{x}, \omega_o) + \int_\Omega f_r(\vec{x}, \omega_i \to \omega_o) W(r(\vec{x}, \omega_i), -\omega_i) \cos\theta_i \, \mathrm{d}\omega_i$$

$$I(\vec{x}, \omega) = \int_{\mathcal{M}} \int_\Omega W_e(\vec{x}, \omega) L_i(\vec{x}, \omega) \cos\theta \, \mathrm{d}\omega \, \mathrm{d}A_{\vec{x}}$$

$$L(\vec{x} \to \vec{x'}) \equiv L(\vec{x'}, \omega_i);\ L(\vec{x'} \to \vec{x''}) \equiv L(\vec{x'}, \omega_o);\ f_r(\vec{x} \to \vec{x'} \to \vec{x''}) \equiv f_r(\vec{x'}, \omega_i \to \omega_o)$$

$$L(\vec{x'} \to \vec{x''}) = L_e(\vec{x'} \to \vec{x''}) + \int_{\mathcal{M}} f_r(\vec{x} \to \vec{x'} \to \vec{x''}) L(\vec{x} \to \vec{x'}) G(\vec{x} \leftrightarrow \vec{x'}) \, \mathrm{d}A_{\vec{x}}$$

$I_j = \int_{\mathcal{M}} \int_{\mathcal{M}} W_e^j(\vec{x} \to \vec{x'}) L(\vec{x} \to \vec{x'}) G(\vec{x} \to \vec{x'}) \, \mathrm{d}A_{\vec{x}} \, \mathrm{d}A_{\vec{x'}}$ for sensor j

$I_j = \int_{\mathcal{P}} f_j(\bar{x}) \, \mathrm{d}\mu(\bar{x})$ where $\bar{x} = x_0 x_1 \cdots x_k$ and

$$f_j(\bar{x}) = L_e(x_0 \to x_1) G(x_0 \to x_1) \prod_{i=1}^{k-1} \left[(f_r(x_{i-1} \to x_i \to x_{i+1}) G(x_i \to x_{i+1})\right] W_e^j(x_{k-1} \to x_k)$$

$\langle I_j \rangle = \frac{f_j(\bar{x})}{p(\bar{x})}$ where $p(\bar{x}) = p(x_0 \ldots x_k) = \prod_{i=0}^{k} p(x_i)$

$$F^{comb} = \sum_{s \geq 0} \sum_{t \geq 0} w_{s,t}(\bar{x}_{s,t}) \frac{f_j(\bar{x}_{s,t})}{p_{s,t}(\bar{x}_{s,t})} \text{ where } w_{s,t}(\bar{x}_{s,t}) = \frac{p_s^2(\bar{x}_{s,t})}{\sum_{i=0}^{s+t-1} p_i^2(\bar{x}_{s,t})}$$

$$B_i = B_{e,i} + \rho_i \sum_{j=1}^{N} B_j \cdot F_{i \to j}$$

$$F_{i \to j} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{V(\vec{x} \leftrightarrow \vec{y}) G(\vec{x} \leftrightarrow \vec{y})}{\pi} \, \mathrm{d}A_j \, \mathrm{d}A_i \text{ where } G(\vec{y} \leftrightarrow \vec{x}) = \frac{\cos\theta_i \cos\theta_j}{\|\vec{x} - \vec{y}\|^2}$$

$$E_i \equiv B_{e,i}$$

$$\begin{bmatrix} 1 - \rho_1 F_{1,1} & -\rho_1 F_{1,2} & \cdots & -\rho_1 F_{1,N} \\ -\rho_2 F_{2,1} & 1 - \rho_2 F_{2,2} & \cdots & -\rho_2 F_{2,N} \\ \cdots & \cdots & & \cdots \\ -\rho_N F_{N,1} & -\rho_N F_{N,2} & \cdots & 1 - \rho_N F_{N,N} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ \cdots \\ B_N \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ \cdots \\ E_N \end{bmatrix}$$

$$F_{i \to j} \cong \sum_{q \in J} \Delta F_q$$

$$\Delta F_1 = \frac{\Delta A}{\pi(x_1^2 + y_1^2 + 1)^2}, \ \Delta F_2 = \frac{z_2 \Delta A}{\pi(x_2^2 + z_2^2 + 1)^2}$$