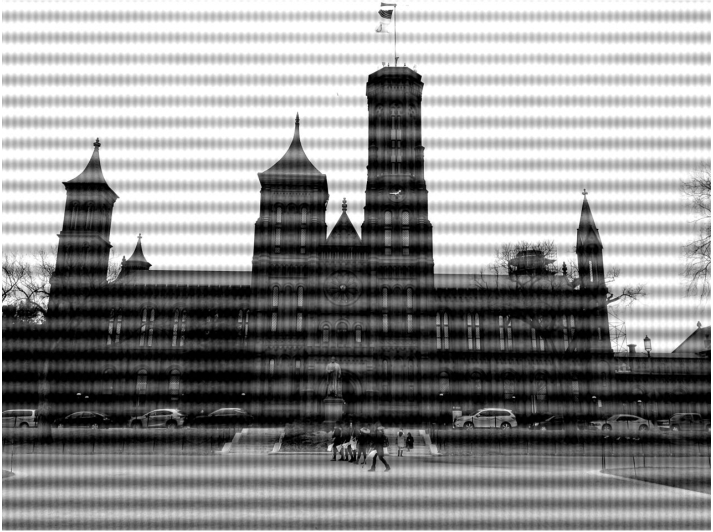
Image preprocessing

Frequency analysis and filtering



Fourier series

Jean Baptiste Joseph Fourier (1768–1830)

- 1822—The Analytical Theory of Heat
- Any function of a variable can be expanded in a series of sines of multiples of the variable

$$f(x) = \sum_{n=0}^{\infty} A_n \sin(B_n x + C_n) =$$

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(B_n x) + b_n \sin(B_n x)$$

Fourier series

- For T-periodic function f, which is integrable on interval of length T

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{T}x\right) + b_n \sin\left(\frac{2\pi n}{T}x\right)$$

- Functions $\sin$ and $\cos$ are orthogonal, i.e.

$$\int_T \sin\left(\frac{2\pi n}{T}x\right) \cos\left(\frac{2\pi m}{T}x\right) dx = 0$$

Fourier series

$$\int_T \cos\left(\frac{2\pi n}{T}x\right)\cos\left(\frac{2\pi m}{T}x\right)dx = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases}$$

similarly for $\sin$

$$\int_T \sin\left(\frac{2\pi n}{T}x\right)\sin\left(\frac{2\pi m}{T}x\right)dx = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \end{cases}$$

This admits computation of coefficients a_n and b_n.

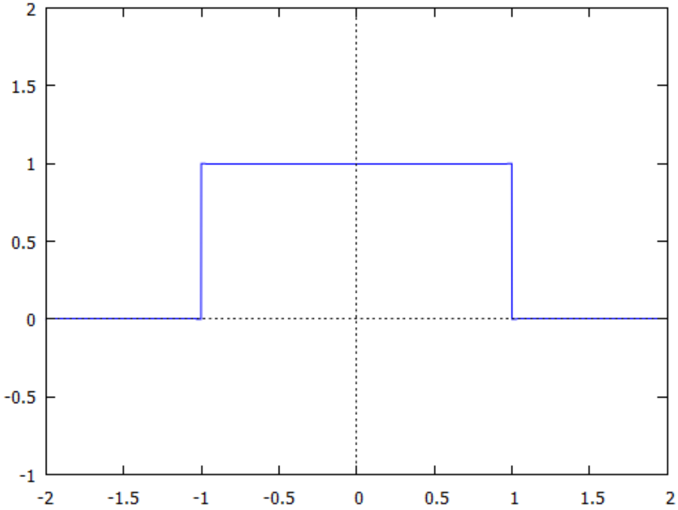
Fourier coefficients

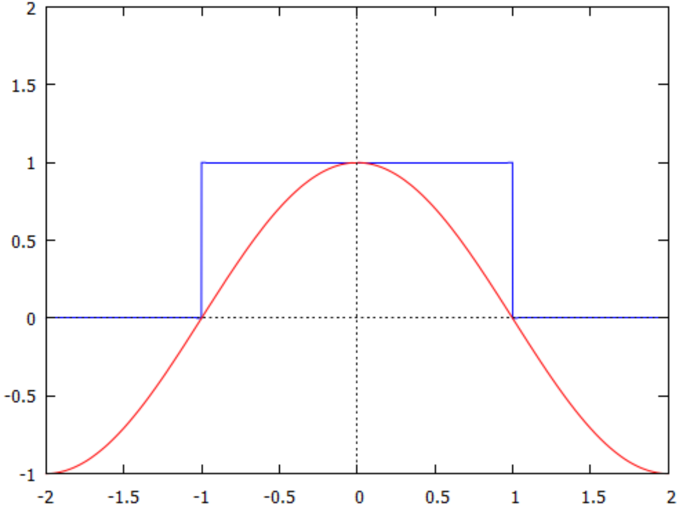
$$a_0 = \frac{1}{T}\int_T f(x)dx,$$
$$a_n = \frac{2}{T}\int_T f(x)\cos\left(\frac{2\pi n}{T}x\right)dx,\ n \neq 0$$
$$b_n = \frac{2}{T}\int_T f(x)\sin\left(\frac{2\pi n}{T}x\right)dx,$$

Geometric viewpoint

- a_0 represents mean value of function f on a given interval
- a_n and b_n represent how much functions $\sin$ and $\cos$ of given frequencies correlate with function f.

Geometric viewpoint

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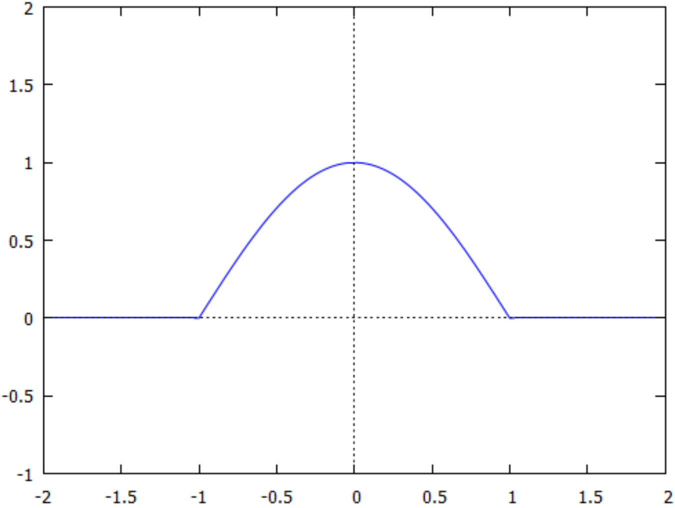
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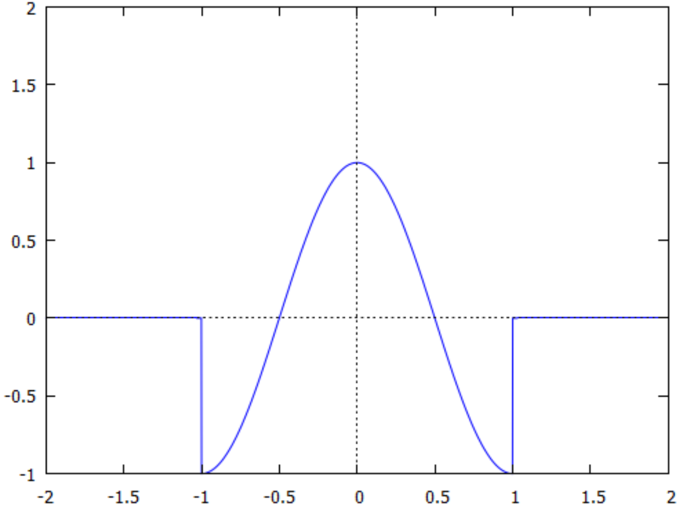
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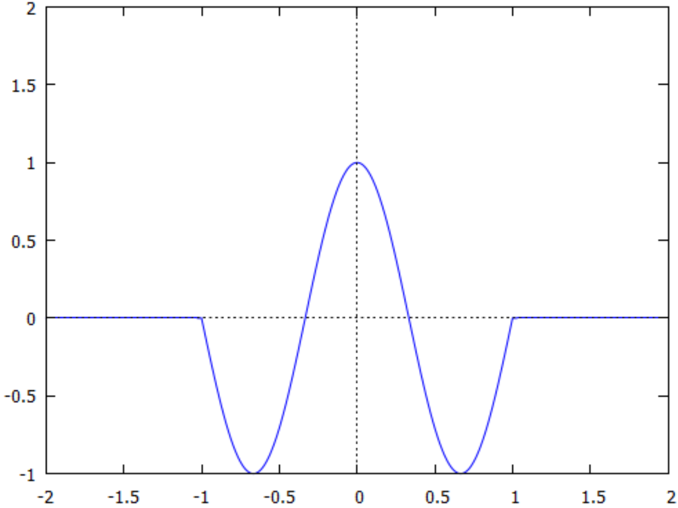

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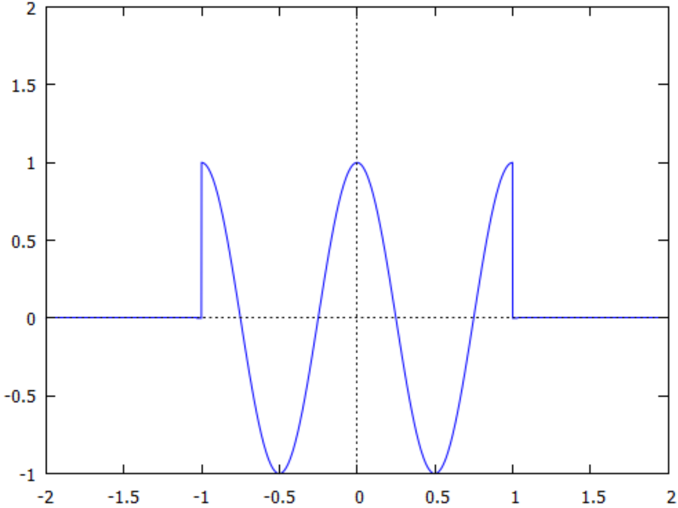
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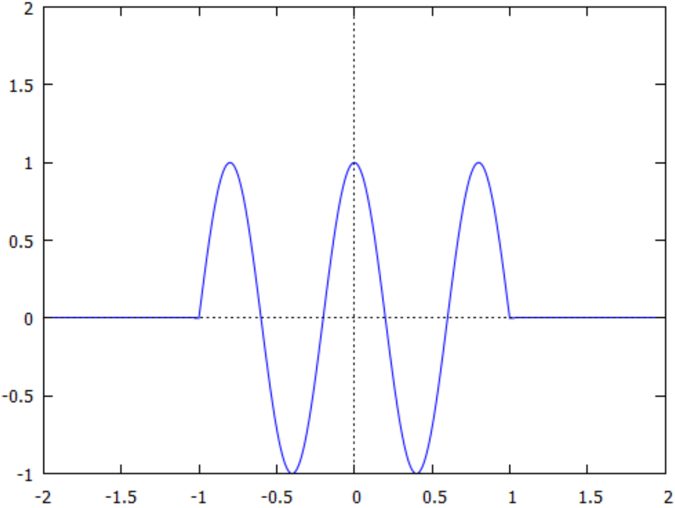

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Complex version

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{T} ix}$$

- Suggested by Euler's formula

$$e^{ix} = \cos x + i \sin x$$

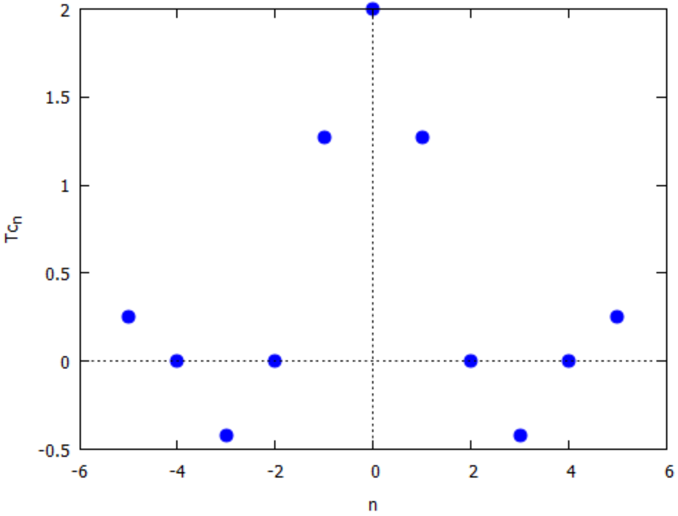
- c_n is in general a complex number.
-

$$c_n = \frac{1}{T} \int_T f(x) e^{-\frac{2\pi n}{T} ix}$$

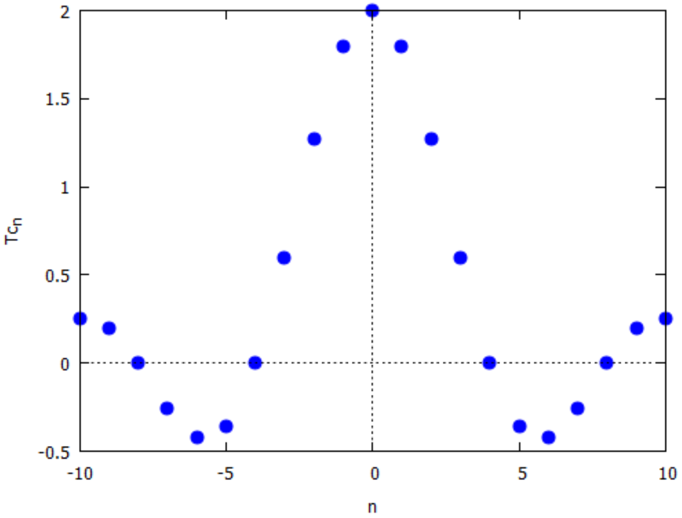
- Assuming f is real, right hand side must be real and c_n must fulfill conditions

$$c_0 = a_0,$$
$$c_n = \frac{a_n - ib_n}{2}, \text{ if } n \geq 1,$$
$$c_n = \bar{c}_{|n|}, \text{ if } n \leq 1.$$

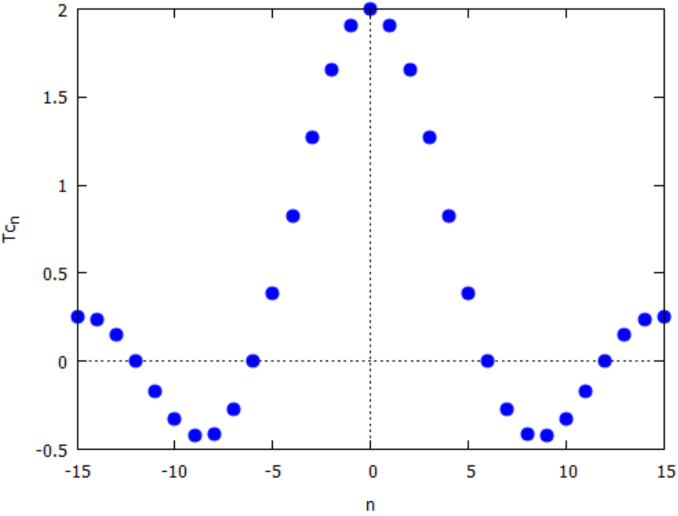
Coefficient plot, $T = 4$



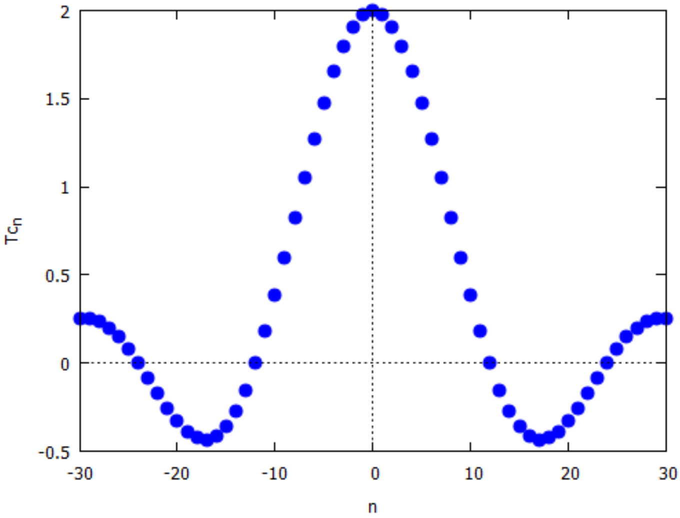
Coefficient plot, $T = 8$



Coefficient plot, $T = 12$



Coefficient plot, $T = 24$



Fourier transform

If $T \to \infty$, for particular frequency $\xi = \frac{n}{T}$ value of integral

$$Tc_n = \int_T f(x) e^{-\frac{2\pi n}{T} ix} dx$$

remains unchanged. We then put

$$\mathcal{F}\{f(x)\} = F(\xi) = Tc_n = \int_{-\infty}^{\infty} f(x) e^{-2\pi \xi ix} dx.$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}\{F(x)\} = f(x) = \int_{-\infty}^{\infty} F(x) e^{2\pi \xi ix} dx.$$

Useful properties

- Linearity

$$\mathcal{F}\{af(x) + bf(y)\} = a\mathcal{F}\{f(x)\} + b\mathcal{F}\{g(x)\}$$

- Duality

$$\mathcal{F}\{F(x)\} = f(-\xi)$$

- Product

Convolution

$$(f * h)(x) \equiv \int_{-\infty}^{\infty} f(x)h(x-\tau)d\tau = \int_{-\infty}^{\infty} f(x-\tau)h(x)d\tau$$

- Expresses amount of overlap of one function $f(x)$ as it is shifted over another function $h(t)$.

Convolution – discrete version

$$f(i,j) = \sum_{(m,n)\in\mathcal{O}} h(i-m, j-n)g(m,n)$$

- $\mathcal{O}$ is a local neighbourhood of pixel (i,j).

Convolution – properties

$$f * h = h * f$$
$$f * (g * h) = (f * g) * h$$
$$f * (g + h) = f * g + f * h$$
$$a(f * g) = (af) * g = f * (ag)$$
$$\frac{d}{dx}(f * h) = \frac{df}{dx} * h = f * \frac{dh}{dx}$$

Back to Fourier

- Product

$$\mathcal{F}\{f(x)g(x)\} = F(\xi) * G(\xi)$$

- Convolution

$$\mathcal{F}\{(f * g)(x)\} = F(\xi)G(\xi)$$

Discreet Fourier transform (DFT)

- Computers deal with discrete signal $f(n)$, $n = 0, \dots N-1$.
- DFT is defined as

$$F(k) = \frac{1}{N} \sum_{n=0}^{N-1} f(n) e^{-2\pi i \frac{nk}{N}}.$$

- Inverse DFT is defined as

$$f(n) = \sum_{k=0}^{N-1} F(k) e^{2\pi i \frac{nk}{N}}.$$

- Spectrum is periodic with period N.

Adding a dimension

- In 2D the Fourier transform can be generalized as

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i(xu+yv)} dx\, dy,$$

- and inverse transform is

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{2\pi i(xu+yv)} du\, dv,$$

Convolution theorem

$$\mathcal{F}\{(f * h)(x,y)\} = F(u,v)H(u,v)$$
$$\mathcal{F}\{(f(x,y)h(x,y)\} = (F * H)(u,v)$$

Discreet version

- DFT in 2D

$$F(u,v) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)e^{-2\pi i\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$

$$u = 0,1,\ldots,M-1, \quad v = 0,1,\ldots,N-1,$$

- and its inverse

$$f(m,n) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{2\pi i\left(\frac{mu}{M}+\frac{nv}{N}\right)}$$

$$m = 0,1,\ldots,M-1, \quad n = 0,1,\ldots,N-1,$$

Different spectra

- Complex spectrum – hard to visualise

$$F(\xi) = \Re(F(\xi)) + i\Im(F(\xi))$$

Suitable for visualisation

- Amplitude spectrum

$$|F(\xi)| = \sqrt{\Re(F(\xi))^2 + \Im(F(\xi))^2}$$

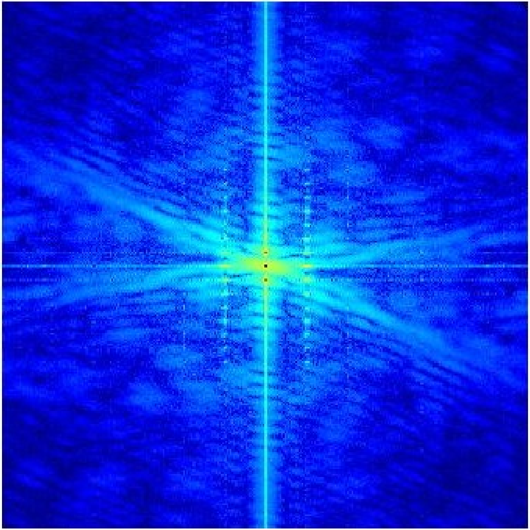
- Phase spectrum

$$\Phi(\xi) = \arctan\left(\frac{\Im(F(\xi))}{\Re(F(\xi))}\right)$$

- Power spectrum

$$P(\xi) = |F(\xi)|^2$$

Amplitude spectrum



Another types of transformations

- Discrete cosine transform
- Wavelet transform

Image sampling

- Dirac distribution $\delta(x, y)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) dx\, dy = 1$$

- Represents ideal impulse

Image sampling

- Sifting property

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-\lambda, y-\mu)dx\,dy = f(\lambda,\mu)$$

Image sampling

- $f(x, y)$ is continuous image function.
- Samples at $x = j\Delta x$, $y = k\Delta y$, $j = 1, \ldots, M$, $K = 1, \ldots, M$.
- Δx, Δy – sampling intervals, i.e. distance of two neighbouring sampling points in x and y direction respectively.
- $f(j\Delta x, k\Delta y)$ – discrete image function

Image sampling

- Ideal sampling in regular grid:

$$s(x, y) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(x - j\Delta x, y - k\delta y)$$

- Sampled image

$$f_s(x, y) = f(x, y)s(x, y)$$

- What happens at frequency spectrum?

Image sampling

- Fourier transform of f_s

$$F_s(u,v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F\left(x - \frac{m}{\Delta x}, y - \frac{n}{\Delta y}\right)$$

- Fourier transform of sampled image is sum of periodically repeated Fourier transform of continuous image function.

Image sampling

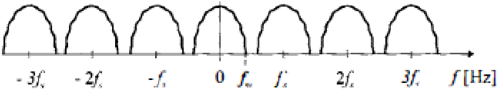


Figure 3.10: Repeated spectra of the 1D signal due to sampling. Non-overlapped case when $f_s \geq 2f_m$.

- f_s is sampling frequency, f_m is maximal frequency in the signal.
- Overlapping means aliasing.
- Shannon sampling theorem

$$\Delta x < \frac{1}{2U}, \qquad \Delta y < \frac{1}{2V},$$

U, V are maximal frequencies.

Image sampling

- Shannon sampling theorem

$$\Delta x < \frac{1}{2U}, \qquad \Delta y < \frac{1}{2V},$$

  U, V are maximal frequencies.
- Sampling interval should be shorter then half of the smallest interesting detail in the image.