

Rotation about an Arbitrary Axis Using Quaternions

Problem. Let $X = (0, -1, 3)^\top$. Rotate X through the angle $\theta = \pi/2$ about the line $p : (2, 1, 2)^\top + t(1, 1, 1)^\top$, $t \in \mathbb{R}$ using quaternions. The positive direction of rotation is determined by the right-hand rule with respect to the direction $(1, 1, 1)^\top$.

Solution

The notation below follows the quaternion rotation operator from the lecture notes. A quaternion is written as

$$\mathbf{q} = q_0 + \vec{q},$$

where q_0 is its scalar part and $\vec{q}$ is its vector part.

1. Translate the line and the point

The rotation formula assumes that the rotation axis passes through the origin. The given line passes through the point

$$P = (2, 1, 2)^\top.$$

Therefore, we translate both the line and the point by the vector $-P$.

$$\begin{aligned} p &\mapsto p' : (0, 0, 0)^\top + t(1, 1, 1)^\top, \\ X &\mapsto X' = X - P = (0, -1, 3)^\top - (2, 1, 2)^\top = (-2, -2, 1)^\top. \end{aligned}$$

The vector to be rotated in the translated coordinate system is thus

$$\vec{v} := X' = (-2, -2, 1)^\top.$$

When used in a quaternion product, $\vec{v}$ is regarded as the purely imaginary quaternion

$$\vec{v} = (0, -2, -2, 1) = -2i - 2j + k.$$

2. Construct the rotation quaternion

The direction vector of the line is $\vec{u} = (1, 1, 1)^\top$. It must first be normalized. As in the lecture notes, we reuse the symbol $\vec{u}$ for the normalized direction vector:

$$\vec{u} := \frac{\vec{u}}{\|\vec{u}\|} = \frac{(1, 1, 1)^\top}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}(1, 1, 1)^\top = \frac{\sqrt{3}}{3}(1, 1, 1)^\top.$$

The corresponding purely imaginary unit quaternion is

$$\mathbf{q}_u := u_x i + u_y j + u_z k = \frac{\sqrt{3}}{3}(i + j + k).$$

For a rotation through the angle θ , the required unit quaternion and its conjugate are

$$\begin{aligned} \mathbf{q}_{u\theta} &:= \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \mathbf{q}_u = q_0 + \vec{q}, \\ \mathbf{q}_{u\theta}^* &:= \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \mathbf{q}_u = q_0 - \vec{q}. \end{aligned}$$

Since $\theta = \pi/2$, we have $\theta/2 = \pi/4$, and therefore

$$\mathbf{q}_{u\theta} = \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \frac{\sqrt{3}}{3}(i + j + k) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \frac{\sqrt{3}}{3}(i + j + k) = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{6}(i + j + k).$$

Thus,

$$q_0 = \frac{\sqrt{2}}{2}, \quad \vec{q} = \frac{\sqrt{6}}{6}(1, 1, 1)^\top,$$

and

$$\mathbf{q}_{u\theta}^* = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{6}(i + j + k).$$

3. Apply the quaternion rotation operator

The rotation operator from the lecture notes is

$$L_q(\vec{v}) := \mathbf{q}_{\mathbf{u}\theta} \vec{v} \mathbf{q}_{\mathbf{u}\theta}^* = \underbrace{(q_0^2 - \|\vec{q}\|^2)\vec{v}}_{(1)} + \underbrace{2(\vec{q} \cdot \vec{v})\vec{q}}_{(2)} + \underbrace{2q_0(\vec{q} \times \vec{v})}_{(3)}.$$

We evaluate its three terms separately.

Term (1): $(q_0^2 - \|\vec{q}\|^2)\vec{v}$

First,

$$q_0^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{1}{2}.$$

Moreover,

$$\|\vec{q}\|^2 = \left(\frac{\sqrt{6}}{6}\right)^2 + \left(\frac{\sqrt{6}}{6}\right)^2 + \left(\frac{\sqrt{6}}{6}\right)^2 = 3 \cdot \frac{6}{36} = \frac{1}{2}.$$

Consequently,

$$(q_0^2 - \|\vec{q}\|^2)\vec{v} = \left(\frac{1}{2} - \frac{1}{2}\right)\vec{v} = (0, 0, 0)^\top.$$

Term (2): $2(\vec{q} \cdot \vec{v})\vec{q}$

The scalar product is:

$$\vec{q} \cdot \vec{v} = \frac{\sqrt{6}}{6}(1, 1, 1) \cdot (-2, -2, 1) = \frac{\sqrt{6}}{6}(-2 - 2 + 1) = \frac{\sqrt{6}}{6}(-3) = -\frac{\sqrt{6}}{2}.$$

Hence,

$$2(\vec{q} \cdot \vec{v})\vec{q} = 2 \left(-\frac{\sqrt{6}}{2}\right) \frac{\sqrt{6}}{6}(1, 1, 1)^\top = -\sqrt{6} \frac{\sqrt{6}}{6}(1, 1, 1)^\top = (-1, -1, -1)^\top.$$

Term (3): $2q_0(\vec{q} \times \vec{v})$

We first calculate the cross product:

$$\vec{q} \times \vec{v} = \frac{\sqrt{6}}{6} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -2 & -2 & 1 \end{vmatrix} = \frac{\sqrt{6}}{6}(3, -3, 0)^\top = \frac{\sqrt{6}}{2}(1, -1, 0)^\top.$$

Therefore,

$$2q_0(\vec{q} \times \vec{v}) = 2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{6}}{2}(1, -1, 0)^\top = \frac{\sqrt{12}}{2}(1, -1, 0)^\top = \sqrt{3}(1, -1, 0)^\top.$$

4. Add the three terms and translate back

Adding terms (1), (2), and (3) gives the rotated vector in the translated coordinate system:

$$\vec{v}_{\text{rot}} = (0, 0, 0)^\top + (-1, -1, -1)^\top + \sqrt{3}(1, -1, 0)^\top = (\sqrt{3} - 1, -\sqrt{3} - 1, -1)^\top.$$

Finally, we undo the initial translation by adding the point $P = (2, 1, 2)^\top$:

$$X_{\text{rot}} = \vec{v}_{\text{rot}} + P = (\sqrt{3} - 1, -\sqrt{3} - 1, -1)^\top + (2, 1, 2)^\top = \boxed{(\sqrt{3} + 1, -\sqrt{3}, 1)^\top}.$$

5. Verification

The translated vector and its rotated image must have the same length:

$$\|\vec{v}\|^2 = (-2)^2 + (-2)^2 + 1^2 = 9, \quad \|\vec{v}_{\text{rot}}\|^2 = (\sqrt{3} - 1)^2 + (-\sqrt{3} - 1)^2 + (-1)^2 = 9.$$

Their components in the direction of the rotation axis are also equal:

$$\vec{u} \cdot \vec{v} = -\sqrt{3} = \vec{u} \cdot \vec{v}_{\text{rot}}.$$

Both checks are consistent with a rigid rotation.